

Likelihood ratio type two-sample tests for current status data

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January 12, 2013

ABSTRACT. We introduce fully nonparametric two-sample tests for testing the null hypothesis that the samples come from the same distribution if the values are only indirectly given via current status censoring. The tests are based on the likelihood ratio principle and allow the observation distributions to be different for the two samples, in contrast with earlier proposals for this situation. A bootstrap method is given for determining critical values and asymptotic theory is developed. A simulation study, using Weibull distributions, is presented to compare the power behavior of the tests with the power of other nonparametric tests in this situation.

Key words: Nonparametric two-sample tests, current status data, maximum smoothed likelihood estimators, likelihood ratio test, Weibull distributions.

1 Introduction

At the beginning of the vast amount of research on right-censored data, there was much interest in two-sample tests for right-censored data, like the Gehan test, log rank test, Efron's test, etc. For example, Gehan (1965) considers the testing problem of testing $F_1 \equiv F_2$ against the alternative $F_1 < F_2$, and gives a permutation test for this testing problem.

Permutation tests for the two-sample problem with interval censored data have been considered in Peto and Peto (1972). Since they rely on the permutation distribution, such tests can only be used when the censoring mechanism is the same in both samples. One of the referees of this paper asked the interesting question whether permutation tests of this type, considered as conditional tests, might be asymptotically independent of the observation distributions in the two samples, in analogy with results in Neuhaus (1993) for two-sample tests in the presence of right censoring. I do not know the answer to this question (current status censoring is very

different from right censoring!), but preliminary results indicate that this method gives very variable estimates of the critical values for moderate sample sizes and therefore cannot be used for these sample sizes. The bootstrap method we propose for computing the critical values does not suffer from this drawback, see section 6.

The maximum likelihood estimator for interval censored data is considered in more detail in Peto (1973), where it is suggested that pointwise standard errors for the survival curve can be estimated from the inverse of the Fisher information. However, we know by now for a long time that this is not correct if we sample from continuous distributions; the pointwise asymptotic distribution is not normal, and the asymptotic variance is not given by the the inverse of the Fisher information, see, e.g., Groeneboom and Wellner (1992) (I owe this observation on Peto (1973) to Peter Sasieni).

Other tests have been considered in, e.g., Andersen and Rønne (1995) and Sun (2006), where also references to earlier work by the latter author can be found. They are based on certain functionals of the distributions which will be different from zero for some alternatives (mostly of the type of “shift alternatives”). Similar tests have been considered in Zhang et al. (2011) and Zhang (2006) for panel count data, where pseudo maximum likelihood estimators are used. Specialized to our present problem, this leads to tests of the same type as the tests in Andersen and Rønne (1995) and Sun (2006).

We consider here rather different types of tests which are likelihood ratio based tests for testing that two samples come from the same distribution, if current status censoring is present. A test of this type is considered in Chapter 3 of Kulikov (2003), where the null hypothesis of equality of the distribution functions F_1 and F_2 , generating the first and second sample, respectively, is tested against Lehmann alternatives of the form

$$F_2(t) = F_1(t)^{1+\theta}, \theta \in (-1, \infty) \setminus \{0\}. \quad (1.1)$$

Here we prefer to test the null hypothesis of equality of F_1 and F_2 just against the more general alternative that they are not equal. Note that in testing against the Lehmann alternatives (1.1), we have to estimate F_1 and θ , whereas in the more general testing problem we have to estimate both F_1 and F_2 nonparametrically.

We will assume the usual conditions for the current status model with continuous distributions, as stated on p. 35 of Groeneboom and Wellner (1992): $(X_1, T_1), \dots, (X_m, T_m)$ and $(X_{m+1}, T_{m+1}), \dots, (X_N, T_N)$, $N = m + n$, are two independent samples of random variables in \mathbb{R}^2 , where X_i and T_i are independent, with, respectively, continuous distribution functions F_1 and G_1 in the first sample and continuous distribution functions F_2 and G_2 in the second sample. We call the X_i the “hidden” variables and the T_i the observation variables. Note that

we allow the distribution functions G_1 and G_2 of the observation variables to be different in the two samples. In the current status model, the only observations which are available to us are the pairs

$$(T_i, \Delta_i), \quad \Delta_i = 1_{\{X_i \leq T_i\}},$$

so we do not observe X_i itself, but only its “current status” Δ_i . In this situation, we want to test the null hypothesis that the distribution functions of the hidden variables are the same in the two samples.

We first discuss what a simple likelihood ratio test would look like. Under the null hypothesis we have to maximize

$$\sum_{i=1}^N \{\Delta_i \log F(T_i) + (1 - \Delta_i) \log (1 - F(T_i))\}, \quad N = m + n,$$

over all distribution functions F , and without the restriction of the null hypothesis we have to maximize

$$\begin{aligned} & \sum_{i=1}^m \{\Delta_i \log F_1(T_i) + (1 - \Delta_i) \log (1 - F_1(T_i))\} \\ & + \sum_{i=m+1}^N \{\Delta_i \log F_2(T_i) + (1 - \Delta_i) \log (1 - F_2(T_i))\} \end{aligned}$$

over all pairs of distribution functions (F_1, F_2) .

This means that under the null hypothesis the MLE (maximum likelihood estimator) is given by the left-continuous slope of the greatest convex minorant of the cusum diagram of the points $(0, 0)$ and the points

$$\left(i, \sum_{j \leq i} \Delta_{(j)} \right), \quad i = 1, \dots, N. \quad (1.2)$$

using a notation, introduced in Groeneboom and Wellner (1992). Here $\Delta_{(j)}$ denotes the indicator corresponding to the j th order statistic $T_{(j)}$. Without the restriction of the null hypothesis the MLE of F_1 is given by the left-continuous slope of the greatest convex minorant of the cusum diagram of the points $(0, 0)$ and the points

$$\left(i, \sum_{j \leq i} \Delta_{(j1)} \right), \quad i = 1, \dots, m, \quad (1.3)$$

where $\Delta_{(j1)}$ is the indicator corresponding to j th order statistic $T_{(j1)}$ of the first sample. Similarly the MLE of F_2 is given by the left-continuous slope of the greatest convex minorant of the cusum diagram of the points $(0, 0)$ and the points

$$\left(i, \sum_{j \leq i} \Delta_{(j2)} \right), \quad i = 1, \dots, n, \quad (1.4)$$

where $\Delta_{(j2)}$ is the indicator corresponding to j th order statistic $T_{(j2)}$ of the second sample.

Let the MLE of F_1 ($= F_2$) under the null hypothesis be given by \hat{F}_N , and let the MLE of the pair (F_1, F_2) without the restriction of the null hypothesis be given by

$$(\hat{F}_{N1}, \hat{F}_{N2}).$$

Then the log likelihood ratio test statistic is given by:

$$\begin{aligned} \sum_{i=1}^m \left\{ \Delta_i \log \frac{\hat{F}_{N1}(T_i)}{\hat{F}_N(T_i)} + (1 - \Delta_i) \log \frac{1 - \hat{F}_{N1}(T_i)}{1 - \hat{F}_N(T_i)} \right\} \\ + \sum_{i=m+1}^N \left\{ \Delta_i \log \frac{\hat{F}_{N2}(T_i)}{\hat{F}_N(T_i)} + (1 - \Delta_i) \log \frac{1 - \hat{F}_{N2}(T_i)}{1 - \hat{F}_N(T_i)} \right\}, \end{aligned} \quad (1.5)$$

where the terms with coefficients Δ_i and $1 - \Delta_i$ are defined to be zero if Δ_i and $1 - \Delta_i$ are zero, respectively.

Although we take this statistic as our inspiration, we first study a statistic somewhat similar to this LR statistic, based on maximum smoothed likelihood estimators (MSLEs), introduced in Groeneboom et al. (2010). One of the reasons is that the asymptotic analysis of the original LR statistic is rather involved; the difficulty in analyzing the limit properties of (1.5) lies in the problem of finding a normalization making it an asymptotic pivot under the null hypothesis. One also has to deal with the non-standard asymptotics, which derives from the fact that the statistic is based on (non-linear) isotonic estimators which satisfy an order restriction. These non-standard features also turn up in the limit behavior. Another reason is that the MSLE leads to more powerful tests for models, commonly used in this type of comparisons. This will be illustrated by a simulation study for a two parameter Weibull distribution, also used in Andersen and Rønn (1995) in a simulation study to check the power of their proposed test.

Maximum smoothed likelihood estimators for current status data were studied in Groeneboom et al. (2010), where it was shown that, under some regularity conditions, the local limit distribution is normal (in contrast with the limit behavior of the original MLE). These estimators are obtained by first smoothing the observation distribution, for example by kernel estimators, and next maximizing the smoothed likelihood w.r.t. the distribution of the hidden variables. In this way the MSLE inherits smoothness properties of the estimate of the observation distribution and converges at a faster rate than the “raw” MLE, which locally converges at rate $n^{-1/3}$ under the usual smoothness conditions on the underlying distributions. Further results on the MSLE can be found in Groeneboom et al. (2010).

A picture of the MSLE estimators and the MLE estimators for samples of size 250 from two different Weibull distributions with densities

$$\alpha_1 \lambda x^{\alpha_1 - 1} e^{-\lambda x^{\alpha_1}}, \quad \alpha_2 \lambda x^{\alpha_2 - 1} e^{-\lambda x^{\alpha_2}}, \quad x > 0, \quad \alpha_1 = 0.5, \alpha_2 = 2, \lambda = 1.6, \quad (1.6)$$

respectively, where $\alpha_1 = 0.5$ holds for the first sample and $\alpha_2 = 2$ for the second sample, is shown in Figure 1.

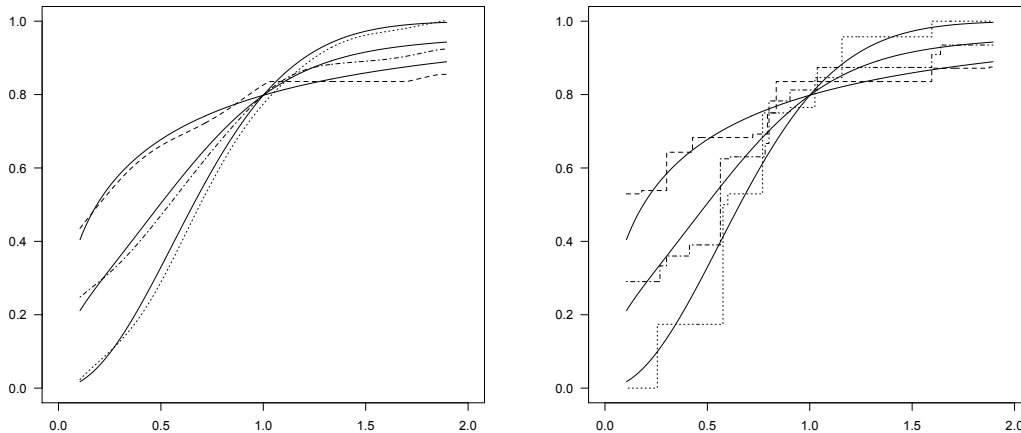


Figure 1: MSLEs and MLEs on $[a, b]$ for samples of size $m = n = 250$ from the Weibull densities (1.6). G_1 and G_2 are uniform on $[0, 2]$, and the interval $[a, b] = [0.1, 1.9]$. The left panel gives the MSLE estimates and the right panel the MLEs, where the dashed curves give the estimates for the first sample ($\alpha_1 = 0.5$), the dotted curves the estimates for the second sample ($\alpha_2 = 2$), and the dashed-dotted curves the estimates for the combined samples. The solid curves give the corresponding actual distribution functions for these three situations. The bandwidth for the computation of the MSLEs was $b_N = 2N^{-1/5} \approx 0.57708$, where $N = m + n = 500$.

2 A likelihood ratio test, based on maximum smoothed likelihood estimators

In order to avoid problems at the boundary, we restrict the domain on which we compute our test statistic to an interval $[a, b] \subset (0, M)$, where $[0, M]$ is assumed to be the support of the underlying densities, corresponding to the distribution functions F_1 and F_2 of the hidden variables. We consider the statistic V_N , similar to (1.5), and defined by

$$V_N = \frac{2m}{N} \int_{t \in [a, b]} \left\{ \tilde{h}_{N1}(t) \log \frac{\tilde{F}_{N1}(t)}{\tilde{F}_N(t)} + \{\tilde{g}_{N1}(t) - \tilde{h}_{N1}(t)\} \log \frac{1 - \tilde{F}_{N1}(t)}{1 - \tilde{F}_N(t)} \right\} dt \\ + \frac{2n}{N} \int_{t \in [a, b]} \left\{ \tilde{h}_{N2}(t) \log \frac{\tilde{F}_{N2}(t)}{\tilde{F}_N(t)} + \{\tilde{g}_{N2}(t) - \tilde{h}_{N2}(t)\} \log \frac{1 - \tilde{F}_{N2}(T_i)}{1 - \tilde{F}_N(T_i)} \right\} dt \quad (2.1)$$

where \tilde{F}_{N1} , \tilde{F}_{N2} and \tilde{F}_N are the maximum smoothed likelihood estimators (MSLEs) for the first, second and combined sample, respectively, and \tilde{g}_{Ni} and \tilde{h}_{Ni} are kernel estimates of the relevant observation densities, defined below. As explained in Groeneboom et al. (2010), where the same type of MSLE for the current status model is defined, the MSLEs for the combined samples and the first and second sample are computed by replacing the cusum diagrams (1.2), (1.3) and (1.4) by the continuous cusum diagrams

$$\left(\tilde{G}_N(t), \tilde{H}_N(t) \right), t \in [0, M], \quad (2.2)$$

$$\left(\tilde{G}_{N1}(t), \tilde{H}_{N1}(t) \right), t \in [0, M], \quad (2.3)$$

and

$$\left(\tilde{G}_{N2}(t), \tilde{H}_{N2}(t) \right), t \in [0, M], \quad (2.4)$$

respectively, where \tilde{G}_N , \tilde{G}_{Ni} , \tilde{H}_N , \tilde{H}_{Ni} and their derivatives are defined in the following way.

We first define the densities \tilde{g}_{Ni} and \tilde{h}_{Ni} on $[b_N, M - b_N]$ by

$$\tilde{g}_{Ni}(t) = \int K_{b_N}(t - u) d\mathbb{G}_{Ni}(u), \quad \tilde{h}_{Ni}(t) = \int K_{b_N}(t - u) \delta d\mathbb{P}_{Ni}(u, \delta), \quad (2.5)$$

Here \mathbb{G}_{N1} is the empirical distribution function of the observations T_1, \dots, T_m of the first sample and \mathbb{P}_{N1} is the empirical distribution of the observations $(T_1, \Delta_1), \dots, (T_m, \Delta_m)$ of the first sample, with the analogous definitions of \mathbb{G}_{N2} and \mathbb{P}_{N2} for the second sample. The densities \tilde{g}_N and \tilde{h}_N are defined on $[b_N, M - b_N]$ by

$$\tilde{g}_N = \alpha_N \tilde{g}_{N1} + \beta_N \tilde{g}_{N2}, \quad \tilde{h}_N = \alpha_N \tilde{h}_{N1} + \beta_N \tilde{h}_{N2}, \quad \alpha_N = \frac{m}{N}, \quad \beta_N = 1 - \alpha_N.$$

The kernel K_b is defined in the usual way by

$$K_b(u) = \frac{1}{b} K(u/b),$$

for a bandwidth $b > 0$, where K is a symmetric positive kernel with compact support. We consider symmetric positive polynomial-type kernels K , with compact support. In our simulation study we took

$$K(u) = \frac{35}{32} (1 - u^2)^3 1_{[-1,1]}(u), \quad (2.6)$$

the so-called triweight kernel.

For $t \in [0, b_N]$ and $t \in [M - b_N, M]$ we use a boundary kernel, defined by a linear combination of $K(u)$ and $uK(u)$. Other ways of bias correction at the boundary are also possible, but it seems necessary to use such a correction in order to obtain a reasonable behavior at the boundary. Using boundary kernels, we lose the simple property that the distribution function can be obtained by just integrating the kernel, and indeed the estimates of the distribution functions

were obtained by numerically integrating the estimates of the densities (and not by integrating the kernels). So we define

$$\begin{aligned}\tilde{G}_{Ni}(t) &= \int_0^t \tilde{g}_{Ni}(u) du, & \tilde{H}_{Ni}(t) &= \int_0^t \tilde{h}_{Ni}(u) du, \\ \tilde{G}_N &= \alpha_N \tilde{G}_{N1} + \beta_N \tilde{G}_{N2}, & \tilde{H}_N &= \alpha_N \tilde{H}_{N1} + \beta_N \tilde{H}_{N2},\end{aligned}$$

and use the corresponding numerical integrals in the continuous cusum diagrams (2.2) to (2.4).

Note that the cusum diagrams (2.2) to (2.4) are continuous analogues of the cusum diagrams (1.2) to (1.4), since, for example, the left-continuous slope of (1.2) is the same as the left-continuous slope of the cusum diagram consisting of the set of points

$$\{(\mathbb{G}_N(t), \mathbb{H}_N(t)), t \geq 0\},$$

where \mathbb{G}_N is the empirical distribution function of the points T_i , $i = 1, \dots, N$, and \mathbb{H}_N is the empirical sub-distribution function of the points T_i , $i = 1, \dots, N$, with $\Delta_i = 1$. However, the slopes of the greatest convex minorants of the continuous cusum diagrams (2.2) to (2.4) are continuous functions of t in contrast with the left-continuous slopes of the cusum diagrams (1.2) to (1.4).

The following result shows that the test statistic V_N is, for a suitable choice of the bandwidth, an asymptotic pivot under the null hypothesis of equality of the two distribution function F_1 and F_2 of the hidden variables in the two samples.

Theorem 2.1 *Let the test statistic V_N be defined by (2.1), using a bandwidth b_N such that $b_N \asymp n^{-\alpha}$, where $2/9 < \alpha < 1/3$. Furthermore, let F stay away from zero and one on $[a, b]$ and have a bounded continuous second derivative f' on an interval (a', b') containing $[a, b]$, and let g_1 and g_2 be continuous densities which stay away from zero on $[a, b]$, with continuous bounded second derivatives on the interval (a', b') . Let the log likelihood ratio statistic V_N , based on the MSLEs, be defined by (2.1). Then we have in probability, if the distribution functions of the hidden variables in the two sample are both equal to F and $m/N \rightarrow \alpha \in (0, 1)$, as $N \rightarrow \infty$,*

$$N \sqrt{\frac{b_N}{b-a}} \left\{ (V_N | T_1, \dots, T_N) - \frac{b-a}{Nb_N} \int K(u)^2 du \right\} \xrightarrow{\mathcal{D}} N(0, \sigma_K^2), \quad (2.7)$$

where $N(0, \sigma_K^2)$ denotes a normal distribution with mean zero and variance

$$\sigma_K^2 = 2 \int \left\{ \int K(u+v) K(u) du \right\}^2 dv.$$

Remark 2.1 To say that (2.7) holds in probability means that

$$\mathbb{P} \left\{ N \sqrt{\frac{b_N}{b-a}} \left\{ V_N - \frac{b-a}{Nb_N} \int K(u)^2 du \right\} \leq x \mid T_1, \dots, T_N \right\} \xrightarrow{p} \Phi(x),$$

for each $x \in \mathbb{R}$, where Φ is the standard normal distribution function and \xrightarrow{p} denotes convergence in probability.

Remark 2.2 The restriction of the bandwidth to the range $N^{-1/3} \ll b_N \ll N^{-2/9}$ has the following motivation. The condition $b_N \gg N^{-1/3}$ is necessary for having the asymptotic equivalence of the MSLEs to ratios of kernel estimators (see Corollary 3.4 in Groeneboom et al. (2010)), and $b_N \ll N^{-2/9}$ prevents the bias to enter, which causes the asymptotic distribution of V_N to become dependent on the observation densities g_1 and g_2 . The bias term drops out if the observation densities g_1 and g_2 are the same in the two samples.

Nevertheless, we prefer to work with a larger bandwidth, at the cost of introducing a bias term, depending on the underlying distributions, as shown in Theorem 2.2. It turns out that this bias term does not bother us, if we compute the critical values by a bootstrap procedure, to be discussed in section 4. The key to this is that the bias term is estimated automatically in the bootstrap resampling from a smooth estimate of F and that the difference between this estimate of the bias and the bias is sufficiently small, as shown in the proof of Theorem 4.1, so that we can replace it by the deterministic bias in the central limit theorem for the bootstrap test statistic.

Theorem 2.2 *Let the test statistic V_N be defined by (2.1), using a bandwidth b_N such that $b_N \asymp n^{-\alpha}$, where $1/5 < \alpha \leq 2/9$. Furthermore, let F stay away from zero and one on $[a, b]$ and have a bounded continuous second derivative f' on an interval (a', b') containing $[a, b]$, and let g_1 and g_2 be continuous densities which stay away from zero on $[a, b]$, with continuous bounded second derivatives on the interval (a', b') . Let the log likelihood ratio statistic V_N , based on the MSLEs, be defined by (2.1). Then we have in probability, if the distribution functions of the hidden variables in the two sample are both equal to F and $m/N \rightarrow \alpha \in (0, 1)$, as $N \rightarrow \infty$,*

$$\begin{aligned} N \sqrt{\frac{b_N}{b-a}} \left\{ (V_N | T_1, \dots, T_N) - \frac{b-a}{Nb_N} \int K(u)^2 du \right. \\ \left. - \alpha_N \beta_N \int_{t=a}^b \frac{f(t)^2 \{g'_1(t)g_2(t) - g'_2(t)g_1(t)\}^2}{F(t)\{1-F(t)\}\bar{g}_N(t)g_1(t)g_2(t)} dt \left\{ \int u^2 K(u) du \right\}^2 b_N^4 \right\} \\ \xrightarrow{\mathcal{D}} N(0, \sigma_K^2), \end{aligned}$$

where \bar{g}_N is defined by:

$$\bar{g}_N(t) = \alpha_N g_1(t) + \beta_N g_2(t).$$

and $N(0, \sigma_K^2)$ denotes a normal distribution with mean zero and variance σ_K^2 defined as in Theorem 2.1.

Remark 2.3 If $b_N \asymp N^{-1/5}$ the situation becomes even more complicated. If the observation densities g_1 and g_2 are the same, we still get the asymptotic normality result, as shown in the

following theorem. But if the densities g_1 and g_2 are different, extra non-negligible random terms enter because of the presence of the bias term. We will not discuss this further in the present paper.

Theorem 2.3 *Let the test statistic V_N be defined by (2.1), using a bandwidth b_N such that $b_N \asymp n^{-\alpha}$, where $1/5 \leq \alpha < 1/3$. Furthermore, let F stay away from zero and one on $[a, b]$ and have a bounded continuous second derivative f' on an interval (a', b') containing $[a, b]$, and let $g_1 = g_2$ be a continuous density which stays away from zero on $[a, b]$, with a continuous bounded second derivative on the interval (a', b') . Then we have in probability, if the distribution functions of the hidden variables in the two sample are both equal to F and $m/N \rightarrow \alpha \in (0, 1)$, as $N \rightarrow \infty$,*

$$N \sqrt{\frac{b_N}{b-a}} \left\{ (V_N | T_1, \dots, T_N) - \frac{b-a}{Nb_N} \int K(u)^2 du \right\} \xrightarrow{\mathcal{D}} N(0, \sigma_K^2),$$

where $N(0, \sigma_K^2)$ denotes a normal distribution with mean zero and variance σ_K^2 defined as in Theorem 2.1.

Remark 2.4 We used a conditional formulation, since we will use conditional tests in our bootstrap approach, but the convergence in distribution will also hold in Theorems 2.1 to 2.3, if we do not condition on T_1, \dots, T_N .

3 The original LR test

We return to the original LR test, using the MLEs, and confine ourselves to a heuristic discussion, since a complete treatment is still out of our grasp. As in the proof of Theorem 2.1, we have:

$$\begin{aligned} & 2 \int_{[a,b]} \left\{ \hat{F}_{N1}(t) \log \frac{\hat{F}_{N1}(t)}{\hat{F}_N(t)} + \{1 - \hat{F}_{N1}(t)\} \log \frac{1 - \hat{F}_{N1}(t)}{1 - \hat{F}_N(t)} \right\} d\mathbb{G}_{N1}(t) \\ & \sim \int_{[a,b]} \frac{\{\hat{F}_N(t) - \hat{F}_{N1}(t)\}^2}{F(t)\{1 - F(t)\}} dG_1(t), \end{aligned}$$

with a similar relation for the terms involving \hat{F}_{N2} . This motivates the study of integrals of the following type:

$$E \int_a^b \frac{\{\hat{F}_N(x) - F(x)\}^2}{F(x)\{1 - F(x)\}} dG(x).$$

The local limit of the MLE of the combined samples under the null hypothesis, when the observation times T_i in both samples is given by G is given in the following theorem, given on p. 89 of Groeneboom and Wellner (1992).

Theorem 3.1 *Let t_0 be such that $0 < F(t_0), G(t_0) < 1$, and let F and G be differentiable at t_0 , with strictly positive derivatives $f(t_0)$ and $g(t_0)$, respectively. Furthermore, let \hat{F}_N be the MLE of F under the null hypothesis. Then we have, as $N \rightarrow \infty$,*

$$N^{1/3} \{ \hat{F}_N(t_0) - F(t_0) \} / \{ \frac{1}{2} F(t_0)(1 - F(t_0)) f(t_0)/g(t_0) \}^{1/3} \xrightarrow{\mathcal{D}} 2Z, \quad (3.1)$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution, and where Z is the last time where standard two-sided Brownian motion plus the parabola $y(t) = t^2$ reaches its minimum.

From this one can deduce, under the assumptions of Theorem 2.1,

$$N^{1/3} E \int_a^b \frac{N^{2/3} \{ \hat{F}_N(x) - F(x) \}^2}{F(x) \{ 1 - F(x) \}} dG(x) \sim N^{1/3} 4EZ^2 \int_a^b \frac{\{ f(x)^2 g(x) \}^{1/3}}{(4F(x) \{ 1 - F(x) \})^{1/3}} dx, \quad N \rightarrow \infty, \quad (3.2)$$

where Z is defined as in Theorem 3.1. By Table 4 in Groeneboom and Wellner (1992) we have:

$$4EZ^2 \approx 1.05423856.$$

Let K_N be the number of jumps of the MLE on the interval $[a, b]$. Then it follows from Groeneboom (2011) that, again under the assumptions of Theorem 2.1,

$$EK_n \sim cN^{1/3} \int_a^b \frac{\{ f(x)^2 g(x) \}^{1/3}}{(4F(x) \{ 1 - F(x) \})^{1/3}} dx, \quad n \rightarrow \infty. \quad (3.3)$$

for a constant $c > 0$ which is close to 2.1, so we find

$$\frac{4EZ^2}{c} \approx 0.5$$

It is tempting to believe that this ratio is exactly equal to $1/2$, but we have no proof of that. It can also be deduced from Groeneboom (2011) that K_N is asymptotically normal and that, in fact,

$$\frac{K_N - EK_N}{\sqrt{EK_N}} \xrightarrow{\mathcal{D}} N(0, c_2), \quad (3.4)$$

for a universal constant $c_2 > 0$, not depending on the underlying distributions.

The intuitive interpretation of all this is that we have histograms with a random number of cells, where, under the null hypothesis \mathcal{H}_0 , the number of cells has an asymptotic expectation which is proportional to the asymptotic expectation on the right-hand side of (3.2). Note that

$$\sqrt{K_N} \left\{ \frac{2T_N}{K_N} - \frac{4EZ^2}{c} \right\} = \sqrt{EK_N} \left\{ \frac{2T_N}{K_N} - \frac{4EZ^2}{c} \right\} + o_p(1),$$

and that

$$\sqrt{EK_N} \left\{ \frac{2T_N}{K_N} - \frac{4EZ^2}{c} \right\} = \sqrt{EK_N} \left\{ \frac{2T_N}{EK_N} - 1 \right\} + \frac{4EZ^2}{c} \frac{EK_N - K_N}{\sqrt{EK_N}} + o_p(1),$$

where c is as in (3.3). Since

$$\frac{EK_N - K_N}{\sqrt{EK_N}} \xrightarrow{\mathcal{D}} N(0, c_2),$$

where c_2 is defined as in (3.4), it is clear that $\sqrt{K_N}\{2T_N/K_N - 1\}$ is an asymptotic pivot under \mathcal{H}_0 if and only if $\sqrt{EK_N}\{2T_N/EK_N - 1\}$ is an asymptotic pivot under \mathcal{H}_0 .

So the situation is somewhat similar to the situation in section 2, but on the other hand much more complicated because of the fact that the MLEs are in fact histogram-type estimators, where the number of cells of the histograms is random, and because of the fact that the estimators \hat{F}_{N1} , \hat{F}_{N2} , and \hat{F}_N are nonlinear estimators which are also asymptotically nonlinear, which leads to non-standard limit distributions of the pointwise estimators $\hat{F}_{Ni}(t)$ and $\hat{F}_N(t)$, in contrast with the MSLEs $\tilde{F}_{Ni}(t)$ and $\tilde{F}_N(t)$ which have normal limit distributions. Another complication is that \hat{F}_N , \hat{F}_{N1} and \hat{F}_{N2} have jumps at different locations.

Nevertheless we want to include this original LR test in our comparisons and we use the bootstrap method of section 4 for generating critical values for this test.

4 A bootstrap method for determining the critical value

We propose the following method for determining the critical value for testing the null hypothesis that the two samples come from the same distribution for the likelihood ratio test, discussed in section 2.

First compute a MSLE $\tilde{F}_{N, \tilde{b}_N}$ for the combined sample as discussed in section 2, using a bandwidth $\tilde{b}_N \asymp N^{-1/5}$. Then, using the observations T_1, \dots, T_m and T_{m+1}, \dots, T_N of the two samples, generate corresponding bootstrap values $\Delta_1^*, \dots, \Delta_m^*$ and $\Delta_{m+1}^*, \dots, \Delta_N^*$ by letting the Δ_i^* be independent Bernoulli ($\tilde{F}_{N, \tilde{b}_N}(T_i)$) random variables. So in practice we generate quasi-random independent Uniform(0, 1) variables U_i^* by using a random number generator, and let Δ_i^* be equal to 1 if $U_i^* < \tilde{F}_{N, \tilde{b}_N}(T_i)$ and zero otherwise. If the observation distributions, generating T_1, \dots, T_m and T_{m+1}, \dots, T_N , respectively, are different, this structure is preserved in this procedure; in the computation of the MSLEs \tilde{F}_{Nj}^* in the bootstrap samples the estimates \tilde{g}_{Nj} of g_j in the original samples are used, for $j = 1, 2$. Repeating this procedure B times, we obtain B bootstrap values $V_{N,i}^*$, $1 \leq i \leq B$, of the test statistic. The distribution of V_N under the null hypothesis is now approximated by the empirical distribution of these bootstrap values and the critical value at (for example) level 5% by the 95th percentile of this set of bootstrap values $V_{N,i}^*$.

In justifying this method for our test statistic V_N , we use the following theorem.

Theorem 4.1 *Let, under either of the conditions of Theorems 2.1 to 2.3, $\tilde{F}_{N,\tilde{b}_N}$ be the MSLE of F under the null hypothesis, defined by the slope of the cusum diagram (2.2), where the bandwidth \tilde{b}_N satisfies $\tilde{b}_N \asymp N^{-1/5}$. Let V_N^* be defined by*

$$V_N^* = 2 \int_{t \in [a,b]} \left\{ \tilde{h}_{N1}^*(t) \log \frac{\tilde{F}_{N1}^*(t)}{\tilde{F}_N^*(t)} + \{ \tilde{g}_{N1}(t) - \tilde{h}_{N1}^*(t) \} \log \frac{1 - \tilde{F}_{N1}^*(t)}{1 - \tilde{F}_N^*(t)} \right\} dt \\ + 2 \int_{t \in [a,b]} \left\{ \tilde{h}_{N2}^*(t) \log \frac{\tilde{F}_{N2}^*(t)}{\tilde{F}_N^*(t)} + \{ \tilde{g}_{N2}(t) - \tilde{h}_{N2}^*(t) \} \log \frac{1 - \tilde{F}_{N2}^*(T_i)}{1 - \tilde{F}_N^*(T_i)} \right\} dt \quad (4.1)$$

where \tilde{F}_N^* , \tilde{F}_{N1}^* and \tilde{F}_{N2}^* are the MSLEs, computed for the samples $(T_1, \Delta_1^*), \dots, (T_m, \Delta_m^*)$ and $(T_{m+1}, \Delta_{m+1}^*), \dots, (T_N, \Delta_N^*)$, and where the Δ_i^* are Bernoulli ($\tilde{F}_{N,b_N}(T_i)$) random variables, generated in the way described before the statement of this theorem; \tilde{g}_{Ni} and \tilde{h}_{Ni}^* are kernel estimates of the relevant observation densities, just as in section 2, where

$$\tilde{h}_{N1}^*(t) = m^{-1} \sum_{i=1}^m \Delta_i^* K_{b_N}(t - T_i), \quad \tilde{h}_{N2}^*(t) = n^{-1} \sum_{i=m+1}^N \Delta_i^* K_{b_N}(t - T_i).$$

with the same bandwidth b_N as taken in the original samples, and where the densities \tilde{g}_{N1} and \tilde{g}_{N2} are the same as in the original samples.

Then we get under \mathcal{H}_0 that the conditional distribution function of V_N^* , given $(T_1, \Delta_1^*), \dots, (T_N, \Delta_N^*)$, rescaled in the same way as in Theorems 2.1 to 2.3 (depending on the choice of bandwidth b_N and presence or absence of the condition $g_1 = g_2$), converges at each $x \in \mathbb{R}$ in probability to the standard normal distribution function $\Phi(x)$.

The proof of this result is given in the appendix. If the null hypothesis does not hold, we follow the same scheme. The critical value is again determined by first computing the Δ_i^* , using the MSLE $\tilde{F}_{N,\tilde{b}_N}$, based on the combined sample.

For the MLEs of section 3 we follow a similar procedure, although we presently cannot justify this with a result analogous to Theorem 4.1. However, the Δ_i^* 's are computed by using the MSLE $\tilde{F}_{N,\tilde{b}_N}$, based on the original combined sample, using a bandwidth $\tilde{b}_N \asymp N^{-1/5}$, instead of the ordinary MLE for this sample. This seems to work better for the sample sizes we used in the simulations. For these distributions, the MSLE converges at the local rate $N^{-2/5}$ instead of MLE itself, which has local rate $N^{-1/3}$, and this led to a better estimate of the level under the null hypothesis, which was taken to be 0.05. Bootstrap estimates, based on the MLE instead of the MSLE, which we also computed, exhibited a very anti-conservative behavior for certain combinations of the parameters, sometimes leading to estimates of the levels which were twice the intended level.

5 Other nonparametric tests

Most test which have been proposed for this problem are based on a comparison of simple functionals of the Δ_i . Under the assumption that the observation times T_i have the same distribution in the two samples, the following test statistic is proposed in Sun (2006):

$$U_{cw} = \sum_{i=1}^N (Z_i - \bar{Z}) \Delta_i = \beta_N \sum_{i=1}^m \Delta_i - \alpha_N \sum_{i=m+1}^N \Delta_i, \quad \alpha_N = \frac{m}{N}, \quad \beta_N = \frac{n}{N}, \quad (5.1)$$

where we take $Z_i = 1$ if the observation belongs to the first sample and $Z_i = 0$ if the observation belongs to the second sample in the notation of Sun (2006), p. 76, and where $\bar{Z} = \sum_{i=1}^N Z_i / N$.

It is stated in Sun (2006) that the variance of $N^{-1/2}$ times (5.1) is given by the random variable

$$N^{-1} \left\{ \sum_{i=1}^m \beta_N^2 \Delta_i^2 + \sum_{i=m+1}^N \alpha_N^2 \Delta_i^2 \right\}. \quad (5.2)$$

Apart from the fact that the variance then is a random variable, we have more difficulties in interpreting this, since we get, if $\alpha_N \rightarrow \alpha \in (0, 1)$ and $\beta_N \rightarrow \beta = 1 - \alpha$,

$$\begin{aligned} N^{-1} \left\{ \sum_{i=1}^m \beta_N^2 \Delta_i^2 + \sum_{i=m+1}^N \alpha_N^2 \Delta_i^2 \right\} &\xrightarrow{p} \alpha\beta \left\{ \beta \int F(t) dG_1(t) + \alpha \int F(t) dG_2(t) \right\} \\ &= \alpha\beta \int F(t) dG(t), \end{aligned}$$

if $G_1 = G_2 = G$. But the actual variance of $N^{-1/2}$ times (5.1) is given by:

$$\alpha_N \beta_N \int F(t) dG(t) \left\{ 1 - \int F(t) dG(t) \right\}, \quad (5.3)$$

if $G_1 = G_2 = G$. So the proposed estimate of the variance in Sun (2006) will severely overestimate the actual variance, and the proposed normalization will not give a standard normal distribution in the limit, as claimed in Sun (2006).

Also, considering the Z_i as i.i.d. random variables, as in Sun (2006), where Z_i is a Bernoulli random variable with

$$\mathbb{P}\{Z_i = 1\} = \alpha_N, \quad \mathbb{P}\{Z_i = 0\} = \beta_N = 1 - \alpha_N,$$

and where the Z_i are independent of the observation times T_i and the indicators Δ_i , we arrive at (5.3) instead of (5.2) as the approximate variance of $N^{-1/2}$ times (5.1). This is seen in the following way.

We can write, under the null hypothesis that the Δ_i have the same distribution, and also

under the restriction that the observations T_i have the same distribution in the two samples,

$$\begin{aligned} U_{cw} &= \sum_{i=1}^N (Z_i - \bar{Z}) \Delta_i = \sum_{i=1}^N (Z_i - \bar{Z}) (\Delta_i - E\Delta_i) \\ &= \sum_{i=1}^N (Z_i - \alpha_N) (\Delta_i - E\Delta_1) + (\alpha_N - \bar{Z}) \sum_{i=1}^N (\Delta_i - E\Delta_1), \end{aligned} \quad (5.4)$$

using $E\Delta_i = E\Delta_1$ for each i . This yields:

$$N^{-1} \text{var}(U_{cw}) \sim \text{var}((Z_1 - \alpha_N)(\Delta_1 - E\Delta_1)) = \alpha_N \beta_N \int F(t) dG(t) \left\{ 1 - \int F(t) dG(t) \right\},$$

since the second expression on the right-hand side of (5.4) gives a contribution of lower order. So we arrive (not surprisingly) again at (5.3) as an approximation of the variance of $n^{-1/2}U_{cw}$ in the interpretation of the Z_i as i.i.d. random variables, implying that the $\hat{\sigma}_{cw}$ suggested as standardization of the statistic $N^{-1/2}U_{cw}$ in Sun (2006) in the last line of the first paragraph of section 4.2.1.1, has to be replaced by an estimate of the square root of (5.3), also if we consider the Z_i to be random. The mistake of taking (5.2) as an estimate of the variance is probably caused by ignoring the dependence of the terms $(Z_i - \bar{Z})\Delta_i$, caused by \bar{Z} , and treating \bar{Z} as if it were EZ_i . The presence of \bar{Z} actually has a variance diminishing effect.

Putting these difficulties aside, and not using the standardization by the square root of (5.2), we could of course consider the test statistic

$$\tilde{U}_N = N^{-1/2} \left\{ \beta_N \sum_{i=1}^m \Delta_i - \alpha_N \sum_{i=m+1}^N \Delta_i \right\} \quad (5.5)$$

which has expectation zero under the null hypothesis, provided $G_1 = G_2$, and variance (5.3), if $G_1 = G_2 = G$. Then, since the MLE \hat{F}_N , based on the combined samples, satisfies, under some regularity conditions,

$$\int \hat{F}_N(t) d\mathbb{G}_N(t) \xrightarrow{p} \int F(t) dG(t),$$

where F is the limit (mixture) distribution of the combined samples (which is the underlying distribution under \mathcal{H}_0), we could use as test statistic

$$U_N = \frac{\tilde{U}_N}{\hat{\sigma}_N}. \quad (5.6)$$

where \tilde{U}_n is defined by (5.5), and where

$$\hat{\sigma}_N^2 = \alpha_N \beta_N \int \hat{F}_N(t) d\mathbb{G}_N(t) \left\{ 1 - \int \hat{F}_N(t) d\mathbb{G}_N(t) \right\}, \quad (5.7)$$

Then U_N tends to a standard normal distribution under the null hypothesis, if $G_1 = G_2 = G$. We note that in Sun (2006) also a test where $G_1 \neq G_2$ is allowed is discussed, but since this

test is connected to a specific parametric model, it is not a test of the fully nonparametric type we consider here.

Andersen and Rønn (1995) consider a test based on

$$W_N = \frac{\sqrt{N} \int_0^a \{\hat{F}_{N1}(t)^2 - \hat{F}_{N2}(t)^2\} d\mathbb{G}_N(t)}{\sqrt{\frac{4}{\alpha_N \beta_N} \int_0^a \hat{F}_N(t)^3 \{1 - \hat{F}_N(t)\} d\mathbb{G}_N(t)}},$$

on an interval $[0, a]$, where W_N is asymptotically standard normal under the null hypothesis, if $G_1 = G_2$ (note that in their definition of this test statistic, which is denoted by W on p. 325, a factor \sqrt{n} is missing in the numerator). They rely in their proof on the master's thesis Hansen (1911), which, incidentally, was written at Delft University of Technology, and not at the University of Copenhagen, as stated in Andersen and Rønn (1995).

Under the conditions of Theorem 2.1 we have:

$$\frac{\sqrt{N} \int_{[a,b]} \{\hat{F}_{N1}(t)^2 - \hat{F}_{N2}(t)^2\} d\mathbb{G}_N(t)}{\sqrt{\frac{4}{\alpha_N \beta_N} \int_{[a,b]} \hat{F}_N(t)^3 \{1 - \hat{F}_N(t)\} d\mathbb{G}_N(t)}} \xrightarrow{\mathcal{D}} N(0, 1), \quad (5.8)$$

under \mathcal{H}_0 , where $N(0, 1)$ is the standard normal distribution. A sketch of how this result can be derived, roughly using the techniques developed in Hansen (1911), is given in the appendix.

6 A simulation study

In this section we compare the LR test based on the MSLEs and the real LR test with the methods, discussed in the preceding section. In our comparison we use the same Weibull model, which was used in the comparison, given in Andersen and Rønn (1995). In determining the critical levels and the powers of the tests, based on V_N (the test statistic based on the MSLEs) and the LR test, based on the MLEs, we used the method described in section 4, that is, the critical values were determined by (Bernoulli) bootstrapping the Δ_i , using the MSLE $\tilde{F}_{N, \tilde{b}_N}(T_i)$ for the combined samples at the observations T_i , by taking 1000 bootstrap samples and determining the 95th percentile of the bootstrap test statistics, so obtained.

As the bandwidth for smoothing the MLE \hat{F}_N , we used $b_N = 2N^{-1/5}$ in all instances, and we used the kernel (2.6) in computing \tilde{F}_N , as described in section 2. As the observation densities g_1 and g_2 for the observation times T_i we took the uniform densities on $[0, 2]$, just as in Andersen and Rønn (1995). Note that in the simulation study of Andersen and Rønn (1995) $g_1 = g_1$, so we can apply Theorem 2.3. This allowed us to resample from the MSLE \tilde{F}_N , which was also used in the computation of the test statistic for the original samples.

The powers and levels computed below for the test statistics V_N (MSLEs) and the LR statistic, based on the MLEs, are determined by taking 1000 samples from the original distributions

and taking 1000 bootstrap sample from each sample, rejecting the null hypothesis if the value in the original sample was larger than the 950th order statistic of the values obtained in the bootstrap samples. The values given in the tables below represent the fraction of rejections for the 1000 samples from the original distributions. The simulation were carried out using a *C* program, which was written by the author specifically for this analysis.

We also included the estimates, discussed in section 5, where W_N denotes the test statistic of Andersen and Rønn (1995) and U_N denotes the test statistic of Sun (2006), but with the incorrect estimate of the variance (5.2) in Sun (2006) replaced by (5.7). In this case we just took 1.96 as our critical value for the absolute value of the test statistic, since the convergence to the standard normal distribution is reasonably fast for these test statistics under the null hypothesis. In this way one can rather fastly compute tables of this type for these test statistics, which was again done by writing a *C* program for this purpose. The tabled values are again based on 1000 samples from the original (Weibull) distributions.

Using the same parametrization as in Andersen and Rønn (1995), we generated the first sample from the density

$$\alpha_1 \lambda x^{\alpha_1-1} e^{-\lambda x^{\alpha_1}}, x > 0, \quad (6.1)$$

and the second sample from the density

$$\alpha_2 \lambda \theta x^{\alpha_2-1} e^{-\lambda \theta x^{\alpha_2}}, x > 0, \quad (6.2)$$

where $\lambda = 1.6$ or $\lambda = 0.58$, and $\alpha_1 = 0.5, 1.0$ or 2.0 . The value of θ is $1, 1.25$ or 2 . Why these specific values were taken in Andersen and Rønn (1995) is not clear to me, but I take the same values for an easy comparison with the work, reported in their paper. I have to note, though, that for $\alpha_i = 0.5$ the Weibull density is unbounded near zero, and that then the results of Hansen (1911) are not valid on $[0, 2]$, since one of the conditions in her thesis was that this density is bounded on the interval of interest. This is also one of the reasons that the interval $[0, 2]$, used in Andersen and Rønn (1995), was shrunk to $[0.1, 1.9]$ in our simulation study, since the density is bounded on this interval.

To illustrate the effect of different observation distributions in the two samples, we generated the first sample of T_i 's again from the uniform density on $[0, 2]$, but the second sample from the decreasing density

$$g_2(t) = \frac{1}{4}(2-t)^3, t \in [0, 2],$$

see Tables 2 and 4. Note that in this case Theorem 2.3 does not apply, and we would actually have to use Theorem 2.1 or 2.2. Nevertheless, we just proceeded in the same way as for the simulations for the situation $g_1 = g_2$, and Tables 2 and 4 show that the test based on the MSLEs,

where we take $b_N = 2N^{-1/5}$ and compute the critical values using the bootstrap procedure, were rather insensitive to the difference of the observation distributions G_1 and G_2 .

Table 1: Estimated levels. The estimation interval is $[0.1, 1.9]$, and $m = n = 50$; $g_1(t) \equiv \frac{1}{2}$, $g_2(t) \equiv \frac{1}{2}$, $\alpha_1 = \alpha_2$. The intended level is $\alpha = 0.05$.

$g_1 = g_2$	λ, α_i	Under H_0				
$m = n = 50$	1.6, 0.5	1.6, 1.0	1.6, 2.0	0.58, 0.5	0.58, 1.0	0.58, 2.0
SLR test	0.041	0.058	0.045	0.049	0.049	0.059
LR test	0.045	0.051	0.041	0.052	0.046	0.055
U_N	0.050	0.060	0.047	0.054	0.058	0.052
W_N	0.055	0.066	0.087	0.061	0.061	0.072

Table 2: Estimated levels. The estimation interval is $[0.1, 1.9]$, and $m = n = 50$; $g_1(t) \equiv \frac{1}{2}$, $g_2(t) = \frac{1}{4}(2 - t)^3$, $\alpha_1 = \alpha_2$. The intended level is $\alpha = 0.05$.

$g_2(t) = \frac{1}{4}(2 - t)^3$	λ, α_i	Under H_0				
$m = n = 50$	1.6, 0.5	1.6, 1.0	1.6, 2.0	0.58, 0.5	0.58, 1.0	0.58, 2.0
SLR test	0.049	0.051	0.045	0.049	0.049	0.059
LR test	0.051	0.055	0.049	0.044	0.050	0.056
U_N	0.422	0.745	0.950	0.262	0.540	0.885
W_N	0.122	0.108	0.130	0.326	0.302	0.276

Table 3: Estimated levels. The estimation interval is $[0.1, 1.9]$, and $m = n = 250$; $g_1(t) \equiv \frac{1}{2}$, $g_2(t) \equiv \frac{1}{2}$, $\alpha_1 = \alpha_2$. The intended level is $\alpha = 0.05$.

$g_1 = g_2$,	λ, α_i	Under H_0				
$m = n = 250$	1.6, 0.5	1.6, 1.0	1.6, 2.0	0.58, 0.5	0.58, 1.0	0.58, 2.0
SLR test	0.051	0.049	0.052	0.053	0.032	0.040
LR test	0.048	0.049	0.059	0.053	0.045	0.054
U_N	0.050	0.060	0.047	0.054	0.058	0.052
W_N	0.055	0.066	0.087	0.061	0.061	0.072

The results of our experiments can be summarized in the following way. The corrected version of the test statistic discussed in Sun (2006), denoted by U_N here, has almost no power

Table 4: Estimated levels. The estimation interval is $[0.1, 1.9]$, and $m = n = 250$. The intended level is $\alpha = 0.05$; $g_1(t) = \frac{1}{2}$, $g_2(t) = \frac{1}{4}(2 - t)^3$, $\alpha_1 = \alpha_2$.

$g_2(t) = \frac{1}{4}(2 - t)^3$	λ, α_i	Under H_0				
$m = n = 250$	1.6, 0.5	1.6, 1.0	1.6, 2.0	0.58, 0.5	0.58, 1.0	0.58, 2.0
SLR test	0.044	0.050	0.051	0.049	0.044	0.051
LR test	0.045	0.051	0.041	0.052	0.054	0.058
U_N	0.970	1.000	1.000	0.840	0.996	1.000
W_N	0.181	0.135	0.102	0.513	0.491	0.410

Table 5: Powers for different shapes, if $m = n = 50$. The estimation interval is $[0.1, 1.9]$.

$g_1 = g_2$	$\lambda, \alpha_1, \alpha_2$	Different shapes		
$m = n = 50$	1.6, 0.5, 1.0	1.6, 0.5, 2.0	0.58, 0.5, 2.0	0.58, 1.0, 2.0
SLR test	0.174	0.675	0.470	0.207
LR test	0.125	0.533	0.364	0.173
U_N	0.061	0.069	0.045	0.053
W_N	0.062	0.110	0.179	0.146

Table 6: Powers for different shapes, if $m = n = 250$. The estimation interval is $[0.1, 1.9]$.

$g_1 = g_2$	$\lambda, \alpha_1, \alpha_2$	Different shapes		
$m = n = 250$	1.6, 0.5, 1.0	1.6, 0.5, 2.0	0.58, 0.5, 2.0	0.58, 1.0, 2.0
SLR test	0.606	1.000	0.990	0.787
LR test	0.440	1.000	0.974	0.610
U_N	0.076	0.132	0.062	0.076
W_N	0.088	0.112	0.583	0.406

for different shape alternatives of the type shown in Figure 1, even for sample sizes $m = n = 250$. The test proposed by Andersen and Rønn (1995), denoted by W_N , has somewhat more power here, but is clearly also not very good for this type of alternative, as already discussed in Andersen and Rønn (1995) (they call this the “crossing alternatives”, since the distribution functions indeed cross). Both the test based on the MSLEs and the test, based on the MLEs, have more power here. The test, based on W_N , is surprisingly powerful for the alternatives which have the same shape but different baseline hazards, and the test, based on U_N also

Table 7: Powers for different baseline hazards, same shape, if $m = n = 50$. The estimation interval is $[0.1, 1.9]$. The parameters α_i are either both 0.5 or both 2 and $\lambda = 1.6$ or 0.58; $\theta = 1.25, 1.5$ or 2.

$g_1 = g_2$	$\lambda, \alpha_i, \theta$	Different baseline hazards				
$m = n = 50$	1.6, 0.5, 1.25	1.6, 0.5, 1.5	1.6, 0.5, 2	0.58, 2, 1.25	0.58, 2, 1.5	0.58, 2, 2
SLR test	0.138	0.283	0.632	0.091	0.208	0.480
LR test	0.097	0.218	0.498	0.082	0.171	0.342
U_N	0.108	0.198	0.441	0.100	0.151	0.333
W_N	0.147	0.352	1.000	0.103	0.293	0.681

Table 8: Powers for different baseline hazards, same shape, if $m = n = 250$. The estimation interval is $[0.1, 1.9]$. The parameters α_i are either both 0.5 or both 2 and $\lambda = 1.6$ or 0.58; $\theta = 1.25, 1.5$ or 2.

$g_1 = g_2$	$\lambda, \alpha_i, \theta$	Different baseline hazards				
$m = n = 250$	1.6, 0.5, 1.25	1.6, 0.5, 1.5	1.6, 0.5, 2	0.58, 2, 1.25	0.58, 2, 1.5	0.58, 2, 2
SLR test	0.377	0.873	1.000	0.227	0.689	0.995
LR test	0.246	0.728	0.996	0.171	0.505	0.964
U_N	0.324	0.721	0.971	0.200	0.495	0.921
W_N	0.473	0.912	1.000	0.337	0.835	1.000

has more power here. The other tests, based on the MSLEs and MLEs, have also reasonable power here, in particular the test based on the MSLEs. Finally, Tables 2 and 4 show that the observation distributions in the two samples can be different if we use the LR-type tests, in contrast with the other tests, considered here. In fact, it has a disastrous effect for the tests U_N and W_N ; U_N even gives 100% rejection under the null hypothesis for several combinations of the parameters.

As noted in the introduction, one could try to use a permutation distribution approach in estimating the levels of the tests under the null hypothesis, also when the observation distributions are different. This does not seem to make much sense for the tests, based on U_N and W_N , but could possibly be of use for the tests, based on the MSLEs and MLEs. We did some experiments in this direction for the Weibull distributions of the simulation study, with rather bad results for our sample sizes $m = n = 50$ and $m = n = 250$. The general finding is that the test based on the MLEs becomes very conservative, whereas the estimates of the levels

for the tests based on the MSLEs become too variable to be of any use. In the latter case one big difference with the approach using the bootstrapped Δ_i is that for the approach using the permutation distribution, the densities g_1 and g_2 have to be estimated anew for every new permutation of the variables $(T_1, \Delta_1), \dots, (T_N, \Delta_N)$, whereas these estimates can be held fixed in the bootstrap approach. This probably leads to a higher variability of the values of the test statistic under the null hypothesis for the permutation approach, leading to unstable estimates of the levels. However, when the observation distributions are the same in the two samples, the permutation procedure seems to work fine, and then gives the same results as the bootstrap procedure.

As a general rule one can say that the tests, based on U_N or W_N , can only have power if the corresponding moment functionals are different from zero. For U_N this functional is given by

$$\int_a^b \{F_1(t) - F_2(t)\} dG(t), \quad (6.3)$$

and for W_N it is given by

$$\int_a^b \{F_1(t)^2 - F_2(t)^2\} dG(t). \quad (6.4)$$

It is clear that F_1 and F_2 can be very different and still satisfy

$$\int_a^b \{F_1(t) - F_2(t)\} dG(t) = 0, \quad \text{or} \quad \int_a^b \{F_1(t)^2 - F_2(t)^2\} dG(t) = 0$$

and in that case that tests, based on U_N or W_N , respectively, will have no power. The LR tests will not suffer from this drawback, since they involve a Kullback-Leibler type distance, and are locally (for example if one would consider contiguous alternatives) equivalent to the squared L_2 -distance

$$\int_a^b \frac{\{F_1(t) - F(t)\}^2}{F(t)\{1 - F(t)\}} dG_1(t) + \int_a^b \frac{\{F_2(t) - F(t)\}^2}{F(t)\{1 - F(t)\}} dG_2(t), \quad (6.5)$$

where F is the distribution function of the combined sample. Moreover, they allow the observation distributions to be different in the two samples, something the other test also do not allow.

The Weibull alternatives, considered in the simulation study, form a family for which the integrals, corresponding to the statistics U_N and W_N are different under the alternatives, considered there. So for these type alternatives the tests U_N and W_N can be expected to have a power exceeding the level of the test. But if the first sample is generated from a Weibull distribution function F_1 with parameters $\alpha = 0.5$ and $\lambda = 0.7$ and the second sample is generated from a Weibull distribution function F_2 with parameters $\alpha = 1.8153$ and $\lambda = 0.7$, the distribution functions are very different (see Figure 2), although we get:

$$\int_a^b \{F_1(t) - F_2(t)\} dt \approx -1.87 \cdot 10^{-6}, \quad a = 0.1, \quad b = 1.9.$$

Taking again the observations G_1 and G_2 to be uniform on $[0, 2]$, we get that the test based on the MSLE has power 0.993 for this alternative, whereas the tests based on U_N has power 0.048 (which is lower than the level 0.05).

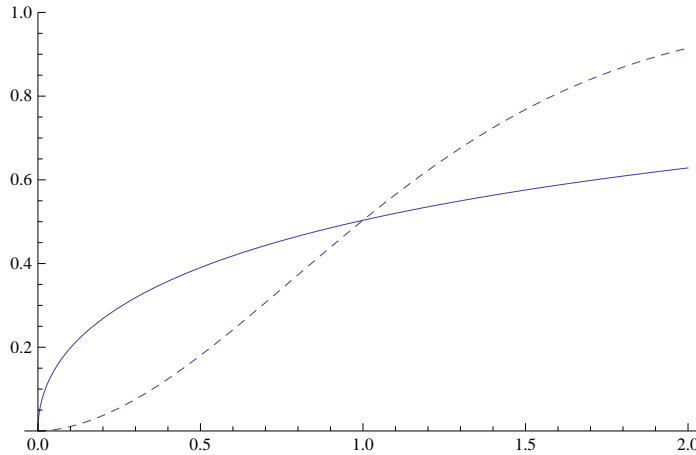


Figure 2: The Weibull distribution function with parameters $\alpha = 0.5$ and $\lambda = 0.7$ (solid curve) and the Weibull distribution function with parameters $\alpha = 1.8153$ and $\lambda = 0.7$ (dashed).

If the first sample is generated from a Weibull distribution function F_1 with parameters $\alpha = 0.2$ and $\lambda = 0.8$ and the second sample is generated from a Weibull distribution function F_2 with parameters $\alpha = 0.767$ and $\lambda = 0.8$, the distribution functions are again rather different (see Figure 3), although we get:

$$\int_a^b \{F_1(t)^2 - F_2(t)^2\} dt \approx 2.6 \cdot 10^{-6}, \quad a = 0.1, \quad b = 1.9.$$

Taking $g_1 = g_2 \equiv (1/2)1_{[0,2]}$ again, the test based on the MSLE has power 0.713 for this alternative, whereas the tests based on W_N has power 0.041 (which is again lower than the level 0.05).

The LR tests, based on the MLEs instead of the MSLEs, has powers 0.964 and 0.515, respectively, for these alternatives, taking the sample sizes $m = n = 250$ again.

7 Concluding remarks

In the preceding, two fully nonparametric tests for the two-sample problem for current status data were discussed. The tests allow the observation distributions for the two samples to be different, and will be consistent for any situation where (6.5) will be different from zero and the distributions satisfy some regularity conditions. For the test, based on the maximum smoothed

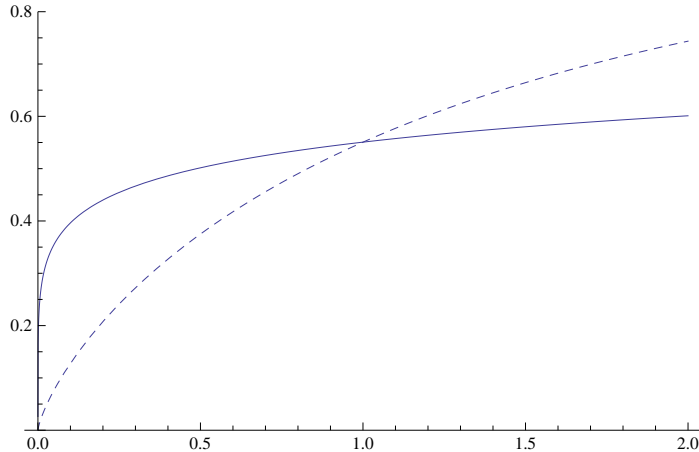


Figure 3: The Weibull distribution function with parameters $\alpha = 0.2$ and $\lambda = 0.8$ (solid curve) and the Weibull distribution function with parameters $\alpha = 0.767$ and $\lambda = 0.8$ (dashed).

likelihood estimators (MSLEs), the theory is more complete than for the test, based on the MLEs, but we suggest a bootstrap method for determining critical values for the latter test, which seemed to work well in the simulation study we conducted.

Most tests which have been proposed for this problem rely on specific functionals, such as (6.3) or (6.4), which can easily be zero, while the distributions F_1 and F_2 are very different. If these functionals are zero, the tests cannot be expected to have power against these alternatives. A simulation study in section 6, using a Weibull model, which was also used in Andersen and Rønn (1995), further illustrates this point.

The convergence to normality in Theorems 2.1 to 2.3 cannot be expected to be very fast. This phenomenon is well-known from the theory of integrated mean squared errors of density estimators. However, the bootstrap procedure we propose for estimating the critical values of the tests, discussed in section 4 seems to work well, even for sample sizes $m = n = 50$. So, for practical purposes, we advise to use this procedure for estimating the critical values of the tests, instead of relying on the asymptotic normality under the null hypothesis.

We have chosen to work with conditional tests, and in this approach we only have to resample the Δ_i in estimating the critical value for the tests. It is also possible to work with unconditional tests, but in that case one also has to resample the T_i from estimates of the densities g_1 and g_2 for the first and second sample, respectively. Preliminary experiments with this procedure indicate that the resulting powers are roughly the same for the model, used in the simulation section 6, but more research is needed to evaluate the two approaches.

8 Appendix

Lemma 8.1 *Let either of the conditions of Theorems 2.1 to 2.3 be satisfied. Then*

$$\sup_{t \in [a, b]} |\tilde{g}_{Nj}(t) - g_j(t)| = O_p \left(N^{-(1-\alpha)/2} \sqrt{\log N} \right) \quad (8.1)$$

and

$$\sup_{t \in [a, b]} \left| \tilde{h}_{Nj}(t) - F(t)g_j(t) \right| = O_p \left(N^{-(1-\alpha)/2} \sqrt{\log N} \right), \quad j = 1, 2, \quad (8.2)$$

implying that also:

$$\sup_{t \in [a, b]} \left| \tilde{F}_{Nj}(t) - F(t) \right| = O_p \left(n^{-(1-\alpha)/2} \sqrt{\log N} \right), \quad j = 1, 2.$$

Proof. By Corollary 3.4 in Groeneboom et al. (2010) we have, with probability tending to one,

$$\tilde{F}_{N1}(t) = \frac{\tilde{h}_{N1}(t)}{\tilde{g}_{N1}(t)}, \quad t \in [a, b], \quad (8.3)$$

that is, the MSLE is just equal to the ratio of two kernel estimators for $t \in [a, b]$, with probability tending to one. Similarly, with probability tending to one,

$$\tilde{F}_{N2}(t) = \frac{\tilde{h}_{N2}(t)}{\tilde{g}_{N2}(t)}, \quad t \in [a, b], \quad (8.4)$$

and

$$\tilde{F}_N(t) = \frac{\alpha_N \tilde{h}_{N1}(t) + \beta_N \tilde{h}_{N2}(t)}{\alpha_N \tilde{g}_{N1}(t) + \beta_N \tilde{g}_{N2}(t)}, \quad t \in [a, b], \quad \alpha_n = m/N, \quad \beta_N = 1 - \alpha_N. \quad (8.5)$$

Hence we assume in the following that \tilde{F}_N , \tilde{F}_{N1} and \tilde{F}_{N2} have the representations (8.3), (8.4) and (8.5), respectively.

We consider the set of functions

$$\mathcal{F} = \left\{ \phi : \phi(x, u \mid t, h) = K \left(\frac{t-u}{h} \right) 1_{[0, u]}(x), \quad t \in [a, b], \quad u \geq 0, \quad h \in (0, c] \right\}, \quad (8.6)$$

where $0 < c \leq (1/2) \min[a, M-b]$, where M is the smallest number such that $\min\{F_1(M), F_2(M)\} = 1$. The kernels, considered in this paper (see section 2) satisfy the condition (K1) of Giné and Guillou (2002), p. 911, implying that \mathcal{F} is a bounded VC class of measurable functions. Furthermore,

$$\begin{aligned} \text{var}(\phi(X_1, T_1 \mid t, h)) &= \text{var} \left(K \left(\frac{t-T_1}{h} \right) \Delta_1 \right) = \int F(u) \{1 - F(u)\} K \left(\frac{t-u}{h} \right)^2 g_1(u) du \\ &= h \int F(t-hw) \{1 - F(t-hw)\} K(w)^2 g_1(t-hw) dw \leq cK(0)^2 \sup_{t \in [a-h, b+h]} g_1(t). \end{aligned}$$

Letting σ and U be defined as in Corollary 2.2 of Giné and Guillou (2002), we get from (2.8) in this corollary the following inequality, based on Talagrand (1994) and Talagrand (1996),

$$\begin{aligned} & \mathbb{P} \left\{ \sup_{t \in [a, b]} \left| \sum_{i=1}^m \left\{ K \left(\frac{t - T_i}{h} \right) \Delta_i - EK \left(\frac{t - T_i}{h} \right) \Delta_i \right\} \right| \geq C\sigma \sqrt{m \log \left(\frac{K(0)}{\sigma} \right)} \right\} \\ & \leq L \exp \left\{ - \frac{C \log \{1 + C/(4L)\}}{L} \log(K(0)/\sigma) \right\}, \end{aligned} \quad (8.7)$$

where L and C are positive constants depending on the VC characteristics of the class \mathcal{F} , and where σ , specialized to our situation, is given by

$$\sigma = K(0) \left(c \sup_{t \in [a-c, b+c]} g_1(t) \right)^{1/2}.$$

Since we take the bandwidth b_N of order $b_N \asymp N^{-\alpha}$, we get from (8.7), taking c in (8.6) also of order $O(n^{-\alpha})$,

$$\begin{aligned} & \sup_{t \in [a, b]} \left| m^{-1} \sum_{i=1}^m K_{b_N}(t - T_i) \Delta_i - EK_{b_N}(t - T_1) \Delta_1 \right| = O_p \left(\sqrt{\frac{1}{Nb_n} \log \frac{K(0)}{\sqrt{b_N}}} \right) \\ & = O_p \left(N^{-(1-\alpha)/2} \sqrt{\log N} \right). \end{aligned}$$

Since we get directly from Theorem 2.3 in Giné and Guillou (2002) that

$$\sup_{t \in [a, b]} |\tilde{g}_{N1}(t) - EK_{b_N}(t - T_1)| = O_p \left(N^{-(1-\alpha)/2} \sqrt{\log N} \right).$$

It now follows from (8.3), which holds with probability tending to one, that also

$$\sup_{t \in [a, b]} \left| \tilde{F}_{N1}(t) - \frac{EK_{b_N}(t - T_1) \Delta_1}{EK_{b_N}(t - T_1)} \right| = O_p \left(N^{-(1-\alpha)/2} \sqrt{\log N} \right).$$

By the conditions of Theorem 2.1 we also have:

$$EK_{b_N}(t - T_1) \Delta_1 = \int K_{b_N}(t - u) F(u) g_1(u) du = F(t) g_1(t) + O(N^{-2\alpha}),$$

and

$$EK_{b_N}(t - T_1) = g_1(t) + O(N^{-2\alpha}),$$

uniformly for $t \in [a, b]$. Hence we obtain:

$$\sup_{t \in [a, b]} |\tilde{F}_{N1}(t) - F(t)| = O_p \left(N^{-(1-\alpha)/2} \sqrt{\log N} \right).$$

The other relations are proved in a similar way. □

Lemma 8.2 *Let either of the conditions of Theorems 2.1 to 2.3 be satisfied. Then*

$$V_n = \alpha_N \beta_N \int_{t \in [a, b]} \frac{\{\tilde{g}_{N2}(t)\tilde{h}_{N1}(t) - \tilde{g}_{N1}(t)\tilde{h}_{N2}(t)\}^2}{F(t)\{1 - F(t)\}\bar{g}_N(t)g_1(t)g_2(t)} dt + O_p\left(N^{-3(1-\alpha)/2}(\log N)^{3/2}\right), \quad (8.8)$$

where

$$\bar{g}_N = \alpha_N g_1 + \beta_N g_2.$$

Moreover,

$$\begin{aligned} & \alpha_N \beta_N \int_{t \in [a, b]} \frac{\{\tilde{g}_{N2}(t)\tilde{h}_{N1}(t) - \tilde{g}_{N1}(t)\tilde{h}_{N2}(t)\}^2}{F(t)\{1 - F(t)\}\bar{g}_N(t)g_1(t)g_2(t)} dt - \frac{b-a}{Nb_N} \int K(u)^2 du \\ &= A_N + B_N - C_N + D_N + o_p\left(\frac{1}{N\sqrt{b_N}}\right), \end{aligned} \quad (8.9)$$

where

$$A_N = \frac{2\alpha_N \beta_N}{m^2} \sum_{1 < i < j \leq m} \{\Delta_i - F(T_i)\}\{\Delta_j - F(T_j)\} \int_{t=a}^b \frac{g_2(t)K_{b_N}(t-T_i)K_{b_N}(t-T_j)}{g_1(t)\bar{g}_N(t)F(t)\{1 - F(t)\}} dt,$$

$$B_N = \frac{2\alpha_N \beta_N}{n^2} \sum_{m < i < j \leq N} \{\Delta_i - F(T_i)\}\{\Delta_j - F(T_j)\} \int_{t=a}^b \frac{g_1(t)K_{b_N}(t-T_i)K_{b_N}(t-T_j)}{g_2(t)\bar{g}_N(t)F(t)\{1 - F(t)\}} dt,$$

$$C_N = \frac{2\alpha_N \beta_N}{mn} \sum_{i=1}^m \sum_{j=m+1}^N \{\Delta_i - F(T_i)\}\{\Delta_j - F(T_j)\} \int_{t=a}^b \frac{K_{b_N}(t-T_i)K_{b_N}(t-T_j)}{\bar{g}_N(t)F(t)\{1 - F(t)\}} dt,$$

and

$$D_N = \alpha_N \beta_N \int_{t=a}^b \frac{f(t)^2 \{g_1'(t)g_2(t) - g_2'(t)g_1(t)\}^2}{F(t)\{1 - F(t)\}\bar{g}_N(t)g_1(t)g_2(t)} dt \left\{ \int u^2 K(u) du \right\}^2 b_N^4.$$

Note that $D_N = 0$ if $g_1 = g_2$.

Proof. By Lemma 8.1 and an expansion of the logarithm we get:

$$\begin{aligned} & 2 \int_{t \in [a, b]} \left\{ \tilde{h}_{N1}(t) \log \frac{\tilde{F}_{N1}(t)}{\tilde{F}_N(t)} + \{\tilde{g}_{N1}(t) - \tilde{h}_{N1}(t)\} \log \frac{1 - \tilde{F}_{N1}(t)}{1 - \tilde{F}_N(t)} \right\} dt \\ &= -2 \int_{t \in [a, b]} \left\{ \tilde{h}_{N1}(t) \log \frac{\tilde{F}_N(t)}{\tilde{F}_{N1}(t)} + \{\tilde{g}_{N1}(t) - \tilde{h}_{N1}(t)\} \log \frac{1 - \tilde{F}_N(t)}{1 - \tilde{F}_{N1}(t)} \right\} dt \\ &= \int_{t \in [a, b]} \frac{\tilde{g}_{N1}(t)^2 \{\tilde{F}_{N1}(t) - \tilde{F}_N(t)\}^2}{\tilde{h}_{N1}(t)\{\tilde{g}_{N1}(t) - \tilde{h}_{N1}(t)\}} \tilde{g}_{N1}(t) dt + O_p\left(N^{-3(1-\alpha)/2}(\log N)^{3/2}\right). \end{aligned}$$

We likewise get, with probability tending to one,

$$\begin{aligned} & 2 \int_{t \in [a, b]} \left\{ \tilde{h}_{N2}(t) \log \frac{\tilde{F}_{N2}(t)}{\tilde{F}_N(t)} + \{\tilde{g}_{N2}(t) - \tilde{h}_{N2}(t)\} \log \frac{1 - \tilde{F}_{N2}(t)}{1 - \tilde{F}_N(t)} \right\} dt \\ &= \int_{t \in [a, b]} \frac{\tilde{g}_{N2}(t)^2 \{\tilde{F}_{N2}(t) - \tilde{F}_N(t)\}^2}{\tilde{h}_{N2}(t)\{\tilde{g}_{N2}(t) - \tilde{h}_{N2}(t)\}} \tilde{g}_{N2}(t) dt + O_p\left(N^{-3(1-\alpha)/2}(\log N)^{3/2}\right). \end{aligned}$$

So we have to consider

$$\begin{aligned}
& \alpha_N \int_{t \in [a, b]} \frac{\tilde{g}_{N1}(t)^2 \{\tilde{F}_{N1}(t) - \tilde{F}_N(t)\}^2}{\tilde{h}_{N1}(t) \{\tilde{g}_{N1}(t) - \tilde{h}_{N1}(t)\}} \tilde{g}_{N1}(t) dt \\
& \quad + \beta_N \int_{t \in [a, b]} \frac{\tilde{g}_{N2}(t)^2 \{F_{N2}(t) - \tilde{F}_N(t)\}^2}{\tilde{h}_{N2}(t) \{\tilde{g}_{N2}(t) - \tilde{h}_{N2}(t)\}} \tilde{g}_{N2}(t) dt \\
& = \alpha_N \beta_N^2 \int_{t \in [a, b]} \frac{\{\tilde{g}_{N2}(t) \tilde{h}_{N1}(t) - \tilde{g}_{N1}(t) \tilde{h}_{N2}(t)\}^2}{\tilde{h}_{N1}(t) \{\tilde{g}_{N1}(t) - \tilde{h}_{N1}(t)\} \tilde{g}_N(t)^2} \tilde{g}_{N1}(t) dt \\
& \quad + \alpha_N^2 \beta_N \int_{t \in [a, b]} \frac{\{\tilde{g}_{N2}(t) \tilde{h}_{N1}(t) - \tilde{g}_{N1}(t) \tilde{h}_{N2}(t)\}^2}{\tilde{h}_{N2}(t) \{\tilde{g}_{N2}(t) - \tilde{h}_{N2}(t)\} \tilde{g}_N(t)^2} \tilde{g}_{N2}(t) dt
\end{aligned}$$

We have:

$$\begin{aligned}
& \frac{\beta_N \tilde{g}_{N1}(t)}{\tilde{h}_{N1}(t) \{\tilde{g}_{N1}(t) - \tilde{h}_{N1}(t)\} \tilde{g}_N(t)^2} + \frac{\alpha_N \tilde{g}_{N2}(t)}{\tilde{h}_{N2}(t) \{\tilde{g}_{N2}(t) - \tilde{h}_{N2}(t)\} \tilde{g}_N(t)^2} \\
& = \frac{\beta_N \tilde{g}_{N1}(t) \tilde{h}_{N2}(t) \{\tilde{g}_{N2}(t) - \tilde{h}_{N2}(t)\} + \alpha_N \tilde{g}_{N2}(t) \tilde{h}_{N1}(t) \{\tilde{g}_{N1}(t) - \tilde{h}_{N1}(t)\}}{\tilde{h}_{N1}(t) \{\tilde{g}_{N1}(t) - \tilde{h}_{N1}(t)\} \tilde{h}_{N2}(t) \{\tilde{g}_{N2}(t) - \tilde{h}_{N2}(t)\} \tilde{g}_N(t)^2} \\
& = \frac{1}{F(t) \{1 - F(t)\} \tilde{g}_N(t) g_1(t) g_2(t)} + O_p \left(N^{-(1-\alpha)/2} \sqrt{\log N} \right).
\end{aligned}$$

Hence:

$$V_n = \alpha_N \beta_N \int_{t \in [a, b]} \frac{\{\tilde{g}_{N2}(t) \tilde{h}_{N1}(t) - \tilde{g}_{N1}(t) \tilde{h}_{N2}(t)\}^2}{F(t) \{1 - F(t)\} \tilde{g}_N(t) g_1(t) g_2(t)} dt + O_p \left(N^{-3(1-\alpha)/2} (\log N)^{3/2} \right).$$

Furthermore,

$$\begin{aligned}
& \tilde{g}_{N2}(t) \tilde{h}_{N1}(t) - \tilde{g}_{N1}(t) \tilde{h}_{N2}(t) \\
& = n^{-1} \sum_{i=m+1}^N K_{b_N}(t - T_i) m^{-1} \sum_{i=1}^m \Delta_i K_{b_N}(t - T_i) \\
& \quad - m^{-1} \sum_{i=1}^m K_{b_N}(t - T_i) n^{-1} \sum_{i=m+1}^n \Delta_i K_{b_N}(t - T_i) \\
& = n^{-1} \sum_{i=m+1}^N K_{b_N}(t - T_i) m^{-1} \sum_{i=1}^m \{F(T_i) - F(t)\} K_{b_N}(t - T_i) \\
& \quad - m^{-1} \sum_{i=1}^m K_{b_N}(t - T_i) n^{-1} \sum_{i=m+1}^n \{F(T_i) - F(t)\} K_{b_N}(t - T_i) \\
& \quad + n^{-1} \sum_{i=m+1}^N K_{b_N}(t - T_i) m^{-1} \sum_{i=1}^m \{\Delta_i - F(T_i)\} K_{b_N}(t - T_i) \\
& \quad - m^{-1} \sum_{i=1}^m K_{b_N}(t - T_i) n^{-1} \sum_{i=m+1}^n \{\Delta_i - F(T_i)\} K_{b_N}(t - T_i).
\end{aligned}$$

We first consider the first two terms on the right-hand side:

$$\begin{aligned}
& n^{-1} \sum_{i=m+1}^N K_{b_N}(t - T_i) m^{-1} \sum_{i=1}^m \{F(T_i) - F(t)\} K_{b_N}(t - T_i) \\
& \quad - m^{-1} \sum_{i=1}^m K_{b_N}(t - T_i) n^{-1} \sum_{i=m+1}^n \{F(T_i) - F(t)\} K_{b_N}(t - T_i) \\
& = \tilde{g}_{N2}(t) m^{-1} \sum_{i=1}^m \{F(T_i) - F(t)\} K_{b_N}(t - T_i) \\
& \quad - \tilde{g}_{N1}(t) n^{-1} \sum_{i=m+1}^n \{F(T_i) - F(t)\} K_{b_N}(t - T_i) \\
& = f(t) \left\{ \tilde{g}_{N2}(t) m^{-1} \sum_{i=1}^m \{T_i - t\} K_{b_N}(t - T_i) - \tilde{g}_{N1}(t) n^{-1} \sum_{i=m+1}^n \{T_i - t\} K_{b_N}(t - T_i) \right\} \\
& \quad + \frac{1}{2} \tilde{g}_{N2}(t) m^{-1} \sum_{i=1}^m f'(\theta_i) \{T_i - t\}^2 K_{b_N}(t - T_i) \\
& \quad - \frac{1}{2} \tilde{g}_{N1}(t) n^{-1} \sum_{i=m+1}^n f'(\theta_i) \{T_i - t\}^2 K_{b_N}(t - T_i),
\end{aligned}$$

where θ_i is a point between t and T_i . This implies, using the fact that the variance is of order $O(N^{-1}b_N)$,

$$\begin{aligned}
& n^{-1} \sum_{i=m+1}^N K_{b_N}(t - T_i) m^{-1} \sum_{i=1}^m \{F(T_i) - F(t)\} K_{b_N}(t - T_i) \\
& \quad - m^{-1} \sum_{i=1}^m K_{b_N}(t - T_i) n^{-1} \sum_{i=m+1}^n \{F(T_i) - F(t)\} K_{b_N}(t - T_i) \\
& = b_N^2 f(t) \{g_1(t)g_2'(t) - g_1'(t)g_2(t)\} \int u^2 K(u) du \\
& \quad + \frac{1}{4} b_N^4 f(t) \{g_1''(t)g_2'(t) - g_1'(t)g_2''(t)\} \left\{ \int u^2 K(u) du \right\}^2 \\
& \quad + \frac{1}{4} b_N^4 f'(t) \{g_1''(t)g_2(t) - g_1(t)g_2''(t)\} \left\{ \int u^2 K(u) du \right\}^2 + O_p \left(\sqrt{\frac{b_N \log N}{N}} \right) + o(b_N^4) \\
& = b_N^2 f(t) \{g_1(t)g_2'(t) - g_1'(t)g_2(t)\} \int u^2 K(u) du + O_p \left(\sqrt{\frac{b_N \log N}{N}} \right) + O(b_N^4), \tag{8.10}
\end{aligned}$$

uniformly for $t \in [a, b]$.

We now define

$$S_N(t) = b_N^2 f(t) \{g_1(t)g_2'(t) - g_1'(t)g_2(t)\} \int u^2 K(u) du. \tag{8.11}$$

and

$$\begin{aligned}
W_N(t) &= \tilde{g}_{N2}(t) m^{-1} \sum_{i=1}^m \{\Delta_i - F(T_i)\} K_{b_N}(t - T_i) \\
&\quad - \tilde{g}_{N1}(t) n^{-1} \sum_{i=m+1}^n \{\Delta_i - F(T_i)\} K_{b_N}(t - T_i).
\end{aligned} \tag{8.12}$$

Then

$$E(W_N(t) \mid T_1, \dots, T_N) = 0, \quad \text{var}(W_N(t)) = O\left(\frac{1}{Nb_N}\right),$$

and hence:

$$\left\{ \tilde{g}_{N2}(t) \tilde{h}_{N1}(t) - \tilde{g}_{N1}(t) \tilde{h}_{N2}(t) \right\}^2 = \{W_N(t) + S_N(t)\}^2 + O_p\left(\frac{\log N}{N}\right) + O_p(b_N^6).$$

We have:

$$\alpha_N \beta_N \int_{t \in [a, b]} \frac{S_N(t) W_N(t)}{F(t) \{1 - F(t)\} \bar{g}_N(t) g_1(t) g_2(t)} dt = O_p\left(\frac{b_N^2}{\sqrt{N}}\right),$$

since, by the central limit theorem

$$\int_{t \in [a, b]} \frac{S_N(t) W_N(t)}{F(t) \{1 - F(t)\} \bar{g}_N(t) g_1(t) g_2(t)} dt = O_p(N^{-1/2}).$$

Note that this term is zero if $g_1 = g_2$.

So we get:

$$\begin{aligned}
&\alpha_N \beta_N \int_{t \in [a, b]} \frac{\left\{ \tilde{g}_{N2}(t) \tilde{h}_{N1}(t) - \tilde{g}_{N1}(t) \tilde{h}_{N2}(t) \right\}^2}{F(t) \{1 - F(t)\} \bar{g}_N(t) g_1(t) g_2(t)} dt \\
&= \alpha_N \beta_N \int_{t \in [a, b]} \frac{W_N(t)^2}{F(t) \{1 - F(t)\} \bar{g}_N(t) g_1(t) g_2(t)} dt + D_N \\
&\quad + O_p\left(\frac{b_N^2}{\sqrt{N}}\right) + O_p\left(\frac{\log N}{N}\right) + O_p(b_N^6),
\end{aligned} \tag{8.13}$$

where D_N is defined as in the formulation of the lemma, and where the term $O_p(b_N^2/\sqrt{N})$ is absent if $g_1 = g_2$. Let

$$W_{N1}(t) = \tilde{g}_{N2}(t) m^{-1} \sum_{i=1}^m \{\Delta_i - F(T_i)\} K_{b_N}(t - T_i),$$

and

$$W_{N2}(t) = \tilde{g}_{N1}(t) n^{-1} \sum_{i=m+1}^n \{\Delta_i - F(T_i)\} K_{b_N}(t - T_i).$$

Then, by definition (8.12), $W_N = W_{N1} + W_{N2}$, and we get:

$$\begin{aligned}
& \alpha_N \beta_N \int_{t \in [a, b]} \frac{W_N(t)^2}{F(t)\{1 - F(t)\}\bar{g}_N(t)g_1(t)g_2(t)} dt \\
&= \alpha_N \beta_N \int_{t \in [a, b]} \frac{W_{N1}(t)^2}{F(t)\{1 - F(t)\}\bar{g}_N(t)g_1(t)g_2(t)} dt \\
&\quad + \alpha_N \beta_N \int_{t \in [a, b]} \frac{W_{N2}(t)^2}{F(t)\{1 - F(t)\}\bar{g}_N(t)g_1(t)g_2(t)} dt \\
&\quad - 2\alpha_N \beta_N \int_{t \in [a, b]} \frac{W_{N1}(t)W_{N2}(t)}{F(t)\{1 - F(t)\}\bar{g}_N(t)g_1(t)g_2(t)} dt
\end{aligned}$$

We now have, using Lemma 8.1 for \tilde{g}_{N2} ,

$$\begin{aligned}
& \alpha_N \beta_N \int_{t \in [a, b]} \frac{W_{N1}(t)^2}{F(t)\{1 - F(t)\}\bar{g}_N(t)g_1(t)g_2(t)} dt \\
&= \alpha_N \beta_N \int_{t \in [a, b]} \frac{g_2(t) \left\{ m^{-1} \sum_{i=1}^m K_{b_N}(t - T_i) \{\Delta_i - F(T_i)\} \right\}^2}{F(t)\{1 - F(t)\}\bar{g}_N(t)g_1(t)} dt + o_p \left(\frac{1}{N\sqrt{b_N}} \right) \\
&= \frac{\alpha_N \beta_N}{m^2} \sum_{i=1}^m \{\Delta_i - F(T_i)\}^2 \int_{t=a}^b \frac{g_2(t) K_{b_N}(t - T_i)^2}{g_1(t)\bar{g}_N(t)F(t)\{1 - F(t)\}} dt \\
&\quad + \frac{2\alpha_N \beta_N}{m^2} \sum_{1 < i < j \leq m} \{\Delta_i - F(T_i)\} \{\Delta_j - F(T_j)\} \int_{t=a}^b \frac{g_2(t) K_{b_N}(t - T_i) K_{b_N}(t - T_j)}{g_1(t)\bar{g}_N(t)F(t)\{1 - F(t)\}} dt \\
&\quad + o_p \left(\frac{1}{N\sqrt{b_N}} \right).
\end{aligned}$$

Moreover, by the central limit theorem,

$$\begin{aligned}
& \frac{\alpha_N \beta_N}{m^2} \sum_{i=1}^m \{\Delta_i - F(T_i)\}^2 \int_{t=a}^b \frac{g_2(t) K_{b_N}(t - T_i)^2}{g_1(t)\bar{g}_N(t)F(t)\{1 - F(t)\}} dt \\
&\quad - \frac{\alpha_N \beta_N}{m^2} E \sum_{i=1}^m \{\Delta_i - F(T_i)\}^2 \int_{t=a}^b \frac{g_2(t) K_{b_N}(t - T_i)^2}{g_1(t)\bar{g}_N(t)F(t)\{1 - F(t)\}} dt \\
&= O_p \left(\frac{1}{N^{3/2}b_N} \right) = o_p \left(\frac{1}{N\sqrt{b_N}} \right),
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\alpha_N \beta_N}{m^2} E \sum_{i=1}^m E \{\Delta_i - F(T_i)\}^2 \int_{t=a}^b \frac{g_2(t) K_{b_N}(t - T_i)^2}{g_1(t)\bar{g}_N(t)F(t)\{1 - F(t)\}} dt \\
&= \frac{\beta_N}{Nm} \sum_{i=1}^m E \left\{ \{\Delta_i - F(T_i)\}^2 \int_{t=a}^b \frac{g_2(t) K_{b_N}(t - T_i)^2}{g_1(t)\bar{g}_N(t)F(t)\{1 - F(t)\}} dt \right\} \\
&= \frac{\beta_N}{N} E \int_{t=a}^b \frac{g_2(t) K_{b_N}(t - T_1)^2}{g_1(t)\bar{g}_N(t)} dt = \frac{1}{Nb_N} \int_{t=a}^b \frac{\beta_N g_2(t)}{\bar{g}_N(t)} dt \int K(u)^2 du + O \left(\frac{b_N}{N} \right).
\end{aligned}$$

We similarly get:

$$\begin{aligned}
& \frac{\alpha_N \beta_N}{n^2} \sum_{i=m+1}^N \{\Delta_i - F(T_i)\}^2 \int_{t=a}^b \frac{g_1(t) K_{b_N}(t - T_i)^2}{g_2(t) \bar{g}_N(t) F(t) \{1 - F(t)\}} dt \\
& \quad - \frac{\alpha_N \beta_N}{n^2} E \sum_{i=m+1}^N \{\Delta_i - F(T_i)\}^2 \int_{t=a}^b \frac{g_1(t) K_{b_N}(t - T_i)^2}{g_2(t) \bar{g}_N(t) F(t) \{1 - F(t)\}} dt \\
& = O_p \left(\frac{1}{N^{3/2} b_N} \right) = o_p \left(\frac{1}{N \sqrt{b_N}} \right),
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\alpha_N \beta_N}{m^2} E \sum_{i=m+1}^N \{\Delta_i - F(T_i)\}^2 \int_{t=a}^b \frac{g_2(t) K_{b_N}(t - T_i)^2}{g_1(t) \bar{g}_N(t) F(t) \{1 - F(t)\}} dt \\
& = \frac{1}{N b_N} \int_{t=a}^b \frac{\alpha_N g_1(t)}{\bar{g}_N(t)} dt \int K(u)^2 du + O \left(\frac{b_N}{N} \right).
\end{aligned}$$

Hence:

$$\begin{aligned}
& \frac{\alpha_N \beta_N}{m^2} \sum_{i=1}^m \{\Delta_i - F(T_i)\}^2 \int_{t=a}^b \frac{g_2(t) K_{b_N}(t - T_i)^2}{g_1(t) \bar{g}_N(t) F(t) \{1 - F(t)\}} dt \\
& \quad + \frac{\alpha_N \beta_N}{n^2} \sum_{i=m+1}^N \{\Delta_i - F(T_i)\}^2 \int_{t=a}^b \frac{g_1(t) K_{b_N}(t - T_i)^2}{g_2(t) \bar{g}_N(t) F(t) \{1 - F(t)\}} dt \\
& = \frac{1}{N b_N} \int_{t=a}^b \frac{\alpha_N g_1(t) + \beta_N g_2(t)}{\bar{g}_N(t)} dt \int K(u)^2 du + o_p \left(\frac{1}{N \sqrt{b_N}} \right) \\
& = \frac{b-a}{N b_N} \int K(u)^2 du + o_p \left(\frac{1}{N \sqrt{b_N}} \right).
\end{aligned}$$

The representation (8.9) now follows. \square

Proofs of Theorem 2.1 to 2.3. By Lemma 8.2, we only have to study the terms on the right-hand side of (8.9). We condition on the values T_1, \dots, T_N . The first term A_N can be written

$$A_N = \sum_{j=2}^m Y_j,$$

where

$$Y_j = \frac{2\alpha_N \beta_N}{m^2} \sum_{1 \leq i < j} \{\Delta_i - F(T_i)\} \{\Delta_j - F(T_j)\} \int_{t=a}^b \frac{g_2(t) K_{b_N}(t - T_i) K_{b_N}(t - T_j)}{g_1(t) \bar{g}_N(t) F(t) \{1 - F(t)\}} dt.$$

Letting \mathcal{F}_j be the σ -algebra, generated by Y_1, \dots, Y_j , and \mathcal{F}_0 be the trivial σ -algebra, we get:

$$E \{Y_j \mid \mathcal{F}_{j-1}\} = 0, \quad j = 1, \dots, m.$$

Furthermore we have, in probability,

$$\begin{aligned}
& E \{ Y_j^2 \mid \mathcal{F}_{j-1} \} \\
&= \frac{4\alpha_N^2 \beta_N^2}{m^4} F(T_j) \{1 - F(T_j)\} \left\{ \sum_{1 \leq i < j} \{ \Delta_i - F(T_i) \} \int_{t=a}^b \frac{g_2(t) K_{b_N}(t - T_i) K_{b_N}(t - T_j)}{g_1(t) \bar{g}_N(t) F(t) \{1 - F(t)\}} dt \right\}^2 \\
&\sim \frac{4(j-1)\alpha_N^2 \beta_N^2}{m^4 b_N} \frac{g_2(T_j)^2 1_{[a-b_N, b+b_N]}(T_j)}{\bar{g}_N(T_j)^2 g_1(T_j)} \int \left\{ \int K(v) K(v+x) dv \right\}^2 dx,
\end{aligned}$$

where the last relation holds for large j . Hence we get, in probability,

$$\begin{aligned}
\sum_{j=1}^m E \{ Y_j^2 \mid \mathcal{F}_{j-1} \} &\sim \sum_{j=1}^m \frac{4(j-1)\beta_N^2}{m^2 N^2 b_N} \frac{g_2(T_j)^2 1_{[a-b_N, b+b_N]}(T_j)}{\bar{g}_N(T_j)^2 g_1(T_j)} \int \left\{ \int K(v) K(v+x) dv \right\}^2 dx \\
&\sim \frac{2m(m-1)\beta_N^2}{m^2 N^2 b_N} \int_a^b \frac{g_2(t)^2}{\bar{g}_N(t)^2} dt \int \left\{ \int K(v) K(v+x) dv \right\}^2 dx \\
&\sim \frac{2\beta_N^2}{N^2 b_N} \int_a^b \frac{g_2(t)^2}{\bar{g}_N(t)^2} dt \int \left\{ \int K(v) K(v+x) dv \right\}^2 dx, \quad m \rightarrow \infty.
\end{aligned}$$

We use here that (for the T_j being random again):

$$\begin{aligned}
& \text{var} \left(\sum_{j=1}^m \frac{4(j-1)}{m^2} \frac{g_2(T_j)^2 1_{[a-b_N, b+b_N]}(T_j)}{\bar{g}_N(T_j)^2 g_1(T_j)} \right) \\
&= \frac{1}{m^4} \text{var} \left(\frac{4g_2(T_1)^2 1_{[a-b_N, b+b_N]}(T_1)}{\bar{g}_N(T_1)^2 g_1(T_1)} \right) \sum_{j=1}^m (j-1)^2 = O\left(\frac{1}{N}\right),
\end{aligned}$$

implying

$$\begin{aligned}
& \sum_{j=1}^m \frac{4(j-1)}{m^2} \frac{g_2(T_j)^2 1_{[a-b_N, b+b_N]}(T_j)}{\bar{g}_N(T_j)^2 g_1(T_j)} \sim \sum_{j=1}^m \frac{4(j-1)}{m^2} E \left\{ \frac{g_2(T_j)^2 1_{[a-b_N, b+b_N]}(T_j)}{\bar{g}_N(T_j)^2 g_1(T_j)} \right\} \\
&\sim \sum_{j=1}^m \frac{4(j-1)}{m^2} \int_a^b \frac{g_2(t)^2}{\bar{g}_N(t)^2} dt \sim \frac{2m(m-1)}{m^2} \int_a^b \frac{g_2(t)^2}{\bar{g}_N(t)^2} dt \sim 2 \int_a^b \frac{g_2(t)^2}{\bar{g}_N(t)^2} dt.
\end{aligned}$$

By similar methods we can extend this to the indices $j = m+1, \dots, N$, where

$$\begin{aligned}
Y_j &= \frac{2\alpha_N \beta_N}{n^2} \sum_{m+1 \leq i < j} \{ \Delta_i - F(T_i) \} \{ \Delta_j - F(T_j) \} \int_{t=a}^b \frac{g_2(t) K_{b_N}(t - T_i) K_{b_N}(t - T_j)}{g_1(t) \bar{g}_N(t) F(t) \{1 - F(t)\}} dt \\
&\quad + \frac{2\alpha_N \beta_N}{mn} \sum_{i=1}^m \{ \Delta_i - F(T_i) \} \{ \Delta_j - F(T_j) \} \int_{t=a}^b \frac{K_{b_N}(t - T_i) K_{b_N}(t - T_j)}{\bar{g}_N(t) F(t) \{1 - F(t)\}} dt,
\end{aligned}$$

which also involves the terms B_N and C_N , and results in:

$$\begin{aligned}
& \sum_{j=1}^N E \{ Y_j^2 \mid \mathcal{F}_{j-1} \} \\
&\sim \frac{2}{N^2 b_N} \left\{ \int K(v) K(v+x) dv \right\}^2 dx \int_{t=a}^b \frac{\beta_N^2 g_2(t)^2 + \alpha_N^2 g_1(t)^2 + 2\alpha_N \beta_N g_1(t) g_2(t)}{\bar{g}_N(t)^2} dt \\
&= \frac{2(b-a)}{N^2 b_N} \left\{ \int K(v) K(v+x) dv \right\}^2 dx.
\end{aligned}$$

So we find:

$$\sum_{j=1}^N E \left\{ \left(N \sqrt{b_N} Y_j \right)^2 \mid \mathcal{F}_{j-1} \right\} \xrightarrow{p} 2(b-a) \left\{ \int K(v) K(v+x) dv \right\}^2 dx, \quad N \rightarrow \infty.$$

By tedious but straightforward computations, using 4th moments of the Bernoulli distribution, one can also check that

$$\sum_{j=1}^N E \left\{ \left(N \sqrt{b_N} Y_j \right)^2 1_{\{N^2 b_N Y_j^2 > \varepsilon\}} \mid \mathcal{F}_{j-1} \right\} \xrightarrow{p} 0, \quad N \rightarrow \infty.$$

The result now follows from the martingale convergence theorem on p. 171 in Pollard (1984). \square

Sketch of proof of (5.8). First consider

$$\int_0^a \left\{ \hat{F}_m(t)^2 - F(t)^2 \right\} dG(t),$$

where we assume $G_1 = G_2 = G$. Then:

$$\begin{aligned} \int_0^a \left\{ \hat{F}_m(t)^2 - F(t)^2 \right\} dG(t) &= 2 \int_0^a \left\{ \hat{F}_m(t) - F(t) \right\} F(t) dG(t) + \int_0^a \left\{ \hat{F}_m(t) - F(t) \right\}^2 dG(t) \\ &= 2 \int_0^a \left\{ \hat{F}_m(t) - F(t) \right\} F(t) dG(t) + O_p \left(m^{-2/3} \right). \end{aligned}$$

Secondly,

$$2 \int_0^a \left\{ \hat{F}_m(t) - F(t) \right\} F(t) dG(t) = 2 \int_0^a \left\{ \hat{F}_m(t) - \delta \right\} F(t) dP_{01}(t, \delta),$$

where P_{01} is the probability measure, generating the random variables $(T_1, \Delta_1), \dots, (T_m, \Delta_m)$.

Let \bar{F} be a piecewise constant version of F , which is constant on the same intervals as \hat{F}_m .

Then:

$$\begin{aligned} &2 \int_0^a \left\{ \hat{F}_m(t) - \delta \right\} F(t) dP_{01}(t, \delta) \\ &= 2 \int_0^a \left\{ \hat{F}_m(t) - \delta \right\} \bar{F}_0(t) dP_{01}(t, \delta) + 2 \int_0^a \left\{ \hat{F}_m(t) - \delta \right\} \{F(t) - \bar{F}_0(t)\} dP_{01}(t, \delta) \\ &= 2 \int_0^a \left\{ \hat{F}_m(t) - \delta \right\} \bar{F}_0(t) dP_{01}(t, \delta) + 2 \int_0^a \left\{ \hat{F}_m(t) - F(t) \right\} \{F(t) - \bar{F}_0(t)\} dG(t) \\ &= 2 \int_0^a \left\{ \hat{F}_m(t) - \delta \right\} \bar{F}_0(t) dP_{01}(t, \delta) + O_p \left(m^{-2/3} \right). \end{aligned}$$

But, by the characterization of the MLE \hat{F}_m , we have, if $\tau(a)$ is the last point of jump of \hat{F}_m before a ,

$$2 \int_{[0, \tau(a))} \left\{ \hat{F}_m(t) - \delta \right\} \bar{F}_0(t) d\mathbb{P}_{N1}(t, \delta) = 0,$$

and hence:

$$\begin{aligned}
2 \int_0^a \{ \hat{F}_m(t) - \delta \} \bar{F}_0(t) dP_{01}(t, \delta) &= 2 \int_{[0, \tau(a))} \{ \hat{F}_m(t) - \delta \} \bar{F}_0(t) d(P_{01} - \mathbb{P}_{N1})(t, \delta) \\
&\quad + O_p(m^{-2/3}) \\
&= 2 \int_{[0, \tau(a))} \{ F(t) - \delta \} \bar{F}_0(t) d(P_{01} - \mathbb{P}_{N1})(t, \delta) \\
&\quad + 2 \int_{[0, \tau(a))} \{ \hat{F}_m(t) - F(t) \} \bar{F}_0(t) d(P_{01} - \mathbb{P}_{N1})(t, \delta) + O_p(m^{-2/3}) \\
&= 2 \int_{[0, a]} \{ F(t) - \delta \} F(t) d(P_{01} - \mathbb{P}_{N1})(t, \delta) + O_p(m^{-2/3}),
\end{aligned}$$

where the first term, multiplied by \sqrt{m} , is asymptotically normal with mean zero and variance

$$4 \int_0^a F(t)^3 \{1 - F(t)\} dG(t).$$

This implies the result, since we can write:

$$\begin{aligned}
&\int_0^a \{ \hat{F}_m(t)^2 - \hat{F}_n(t)^2 \} d\mathbb{G}_N(t) \\
&= \int_0^a \{ \hat{F}_m(t)^2 - \hat{F}_n(t)^2 \} dG(t) + \int_{[0, a]} \{ \hat{F}_m(t)^2 - \hat{F}_n(t)^2 \} d(\mathbb{G}_N - G)(t) \\
&= \int_0^a \{ \hat{F}_m(t)^2 - F(t)^2 \} dG(t) - \int_0^a \{ \hat{F}_n(t)^2 - F(t)^2 \} dG(t) + O_p(N^{-2/3}),
\end{aligned}$$

and since \hat{F}_m and \hat{F}_n are based on independent samples. \square

Proof of Theorem 4.1. We may assume that, for large N , $\tilde{F}_{N, \tilde{b}_N}$ has the representation

$$\tilde{F}_{N, \tilde{b}_N}(t) = \frac{\int \delta K_{\tilde{b}_N}(t-u) d\mathbb{P}_N(u, \delta)}{\int K_{\tilde{b}_N}(t-u) d\mathbb{G}_N(u)}.$$

for $t \in [a, b]$, where $\tilde{b}_N \asymp N^{-1/5}$. This gives

$$\tilde{f}_{N, \tilde{b}_N}(t) = \frac{\int \delta K'_{\tilde{b}_N}(t-u) d\mathbb{P}_N(u, \delta)}{\tilde{g}_{N, \tilde{b}_N}(t)} - \frac{\tilde{g}'_{N, \tilde{b}_N}(t) \int \delta K_{\tilde{b}_N}(t-u) d\mathbb{P}_N(u, \delta)}{\tilde{g}_{N, \tilde{b}_N}(t)^2},$$

where

$$\tilde{g}_{N, \tilde{b}_N}(t) = \int K_{\tilde{b}_N}(t-u) d\mathbb{G}_N(u), \quad \tilde{g}'_{N, \tilde{b}_N}(t) = \int K'_{\tilde{b}_N}(t-u) d\mathbb{G}_N(u),$$

and

$$K'_{\tilde{b}_N}(t-u) = \frac{1}{\tilde{b}_N^2} K' \left(\frac{t-u}{\tilde{b}_N} \right)$$

By the assumptions on g , and using $\tilde{b}_N \asymp n^{-1/5}$, we have

$$\begin{aligned}
\sup_{t \in [a, b]} \left| \tilde{g}_{N, \tilde{b}_N}(t) - \bar{g}_N(t) \right| &= O_p \left(N^{-2/5} \sqrt{\log n} \right) \quad \text{and} \quad \sup_{t \in [a, b]} \left| \tilde{g}'_{N, \tilde{b}_N}(t) - g'(t) \right| \\
&= O_p \left(N^{-1/5} \sqrt{\log n} \right)
\end{aligned}$$

uniformly for $t \in [a, b]$. Furthermore, since

$$\int \delta K'_{\tilde{b}_N}(t-u) d\mathbb{P}_N(u, \delta) = \frac{1}{N\tilde{b}_N^2} \sum_{i=1}^N K' \left(\frac{t-T_i}{\tilde{b}_N} \right) \Delta_i$$

we get:

$$\begin{aligned} & \int \delta K'_{\tilde{b}_N}(t-u) d\mathbb{P}_N(u, \delta) - \int K'_{\tilde{b}_N}(t-u) F(u) dG(u) \\ &= \int \{\delta - F(u)\} K'_{\tilde{b}_N}(t-u) d\mathbb{P}_N(u, \delta) + \int F(u) K'_{\tilde{b}_N}(t-u) d(\mathbb{G}_N - G)(u) \end{aligned}$$

and hence

$$\sup_{t \in [a, b]} \left| \tilde{f}_{N, \tilde{b}_N}(t) - f(t) \right| = O_p \left(N^{-1/5} \sqrt{\log N} \right). \quad (8.14)$$

It can be proved in a similar way that

$$\sup_{x \in [a, b]} \left| \tilde{F}_{N, \tilde{b}_N}(t) - F(t) \right| = O_p \left(N^{-2/5} \sqrt{\log N} \right).$$

The bootstrap test statistic V_N^* now has the representation

$$\begin{aligned} V_N &= \frac{2m}{N} \int_{t \in [a, b]} \left\{ \tilde{h}_{N1}^*(t) \log \frac{\tilde{F}_{N1}^*(t)}{\tilde{F}_N^*(t)} + \{\tilde{g}_{N1}(t) - \tilde{h}_{N1}^*(t)\} \log \frac{1 - \tilde{F}_{N1}^*(t)}{1 - \tilde{F}_N^*(t)} \right\} dt \\ &\quad + \frac{2n}{N} \int_{t \in [a, b]} \left\{ \tilde{h}_{N2}^*(t) \log \frac{\tilde{F}_{N2}^*(t)}{\tilde{F}_N^*(t)} + \{\tilde{g}_{N2}(t) - \tilde{h}_{N2}^*(t)\} \log \frac{1 - \tilde{F}_{N2}^*(T_i)}{1 - \tilde{F}_N^*(T_i)} \right\} dt, \end{aligned}$$

where

$$\tilde{h}_{Nj}^*(t) = \int \delta^* K_{b_N}(t-u) d\mathbb{P}_{Nj}(u, \delta^*), \quad j = 1, 2,$$

and the Δ_i^* are defined by

$$\Delta_i^* = 1_{[0, \tilde{F}_{N, \tilde{b}_N}(T_i)]}(U_i^*),$$

for independent random variables U_1^*, \dots, U_N^* , independent of the random variables (T_i, Δ_i) , $i = 1, \dots, N$, and where we may assume, as before, that

$$\tilde{F}_{Nj}^*(t) = \frac{\int \delta^* K_{b_N}(t-u) d\mathbb{P}_{Nj}(u, \delta^*)}{\tilde{g}_{Nj}(t)}, \quad j = 1, 2.$$

Note that the only extra randomness is introduced by the uniform random variables U_i^* , and that the bandwidth b_N , used here, may be smaller than the bandwidth \tilde{b}_N , used in the computation of $\tilde{F}_{N, \tilde{b}_N}$. In fact, b_N is the bandwidth which is used in the original sample and we have, by assumption

$$b_N \asymp N^{-\alpha},$$

where $1/3 < \alpha < 1/5$, and where we allow $\alpha = 1/5$ if it is assumed that $g_1 = g_2$. The densities \tilde{g}_{Nj} have been computed in the original sample, using this possibly smaller bandwidth b_N .

We now get, similarly as in Lemma 8.2,

$$V_n^* = \alpha_N \beta_N \int_{t \in [a, b]} \frac{\{\tilde{g}_{N2}(t) \tilde{h}_{N1}^*(t) - \tilde{g}_{N1}(t) \tilde{h}_{N2}^*(t)\}^2}{\tilde{F}_{N, \tilde{b}_N}(t) \{1 - \tilde{F}_{N, \tilde{b}_N}(t)\} \tilde{g}_N(t) g_1(t) g_2(t)} dt + o_p \left(\frac{1}{N \sqrt{b_N}} \right),$$

where

$$\bar{g}_N = \alpha_N g_1 + \beta_N g_2,$$

and

$$\begin{aligned} & \tilde{g}_{N2}(t) \tilde{h}_{N1}^*(t) - \tilde{g}_{N1}(t) \tilde{h}_{N2}^*(t) \\ &= n^{-1} \sum_{i=m+1}^N K_{b_N}(t - T_i) m^{-1} \sum_{i=1}^m \Delta_i^* K_{b_N}(t - T_i) \\ & \quad - m^{-1} \sum_{i=1}^m K_{b_N}(t - T_i) n^{-1} \sum_{i=m+1}^n \Delta_i^* K_{b_N}(t - T_i) \\ &= n^{-1} \sum_{i=m+1}^N K_{b_N}(t - T_i) m^{-1} \sum_{i=1}^m \{\tilde{F}_{N, \tilde{b}_N}(T_i) - \tilde{F}_{N, \tilde{b}_N}(t)\} K_{b_N}(t - T_i) \\ & \quad - m^{-1} \sum_{i=1}^m K_{b_N}(t - T_i) n^{-1} \sum_{i=m+1}^n \{\tilde{F}_{N, \tilde{b}_N}(T_i) - \tilde{F}_{N, \tilde{b}_N}(t)\} K_{b_N}(t - T_i) \\ & \quad + n^{-1} \sum_{i=m+1}^N K_{b_N}(t - T_i) m^{-1} \sum_{i=1}^m \{\Delta_i^* - \tilde{F}_{N, \tilde{b}_N}(T_i)\} K_{b_N}(t - T_i) \\ & \quad - m^{-1} \sum_{i=1}^m K_{b_N}(t - T_i) n^{-1} \sum_{i=m+1}^n \{\Delta_i^* - \tilde{F}_{N, \tilde{b}_N}(T_i)\} K_{b_N}(t - T_i). \end{aligned}$$

This is the same decomposition as used in the proof of Lemma 8.2, but with F replaced by $\tilde{F}_{N, \tilde{b}_N}$ and Δ_i replaced by Δ_i^* . Instead of W_N , defined by (8.12), we get:

$$\begin{aligned} W_N^*(t) &= \tilde{g}_{N2}(t) m^{-1} \sum_{i=1}^m \{\Delta_i^* - \tilde{F}_{N, \tilde{b}_N}(T_i)\} K_{b_N}(t - T_i) \\ & \quad - \tilde{g}_{N1}(t) n^{-1} \sum_{i=m+1}^n \{\Delta_i^* - \tilde{F}_{N, \tilde{b}_N}(T_i)\} K_{b_N}(t - T_i). \end{aligned}$$

and

$$\left\{ \tilde{g}_{N2}(t) \tilde{h}_{N1}^*(t) - \tilde{g}_{N1}(t) \tilde{h}_{N2}^*(t) \right\}^2 = \{W_N^*(t) + S_N^*(t)\}^2 + o_p \left(\frac{1}{N \sqrt{b_N}} \right),$$

where

$$S_N^*(t) = b_N^2 \tilde{f}_{N, \tilde{b}_N}(t) \{g_1(t) g_2'(t) - g_1'(t) g_2(t)\} \int u^2 K(u) du.$$

Moreover,

$$\begin{aligned} & \alpha_N \beta_N \int_{t \in [a, b]} \frac{\{\tilde{g}_{N2}(t) \tilde{h}_{N1}^*(t) - \tilde{g}_{N1}(t) \tilde{h}_{N2}^*(t)\}^2}{\tilde{F}_{N, \tilde{b}_N}(t) \{1 - \tilde{F}_{N, \tilde{b}_N}(t)\} \tilde{g}_N(t) g_1(t) g_2(t)} dt - \frac{b-a}{N b_N} \int K(u)^2 du \\ &= A_N^* + B_N^* - C_n^* + D_N + o_p \left(\frac{1}{N \sqrt{b_N}} \right), \end{aligned} \tag{8.15}$$

where

$$\begin{aligned}
A_N^* &= \frac{2\alpha_N\beta_N}{m^2} \sum_{1 < i < j \leq m} \{\Delta_i^* - \tilde{F}_{N,\tilde{b}_N}(T_i)\} \{\Delta_j^* - \tilde{F}_{N,\tilde{b}_N}(T_j)\} \\
&\quad \cdot \int_{t=a}^b \frac{g_2(t)K_{b_N}(t-T_i)K_{b_N}(t-T_j)}{g_1(t)\bar{g}_N(t)\tilde{F}_{N,\tilde{b}_N}(t)\{1-\tilde{F}_{N,\tilde{b}_N}(t)\}} dt, \\
B_N^* &= \frac{2\alpha_N\beta_N}{n^2} \sum_{m < i < j \leq N} \{\Delta_i^* - \tilde{F}_{N,\tilde{b}_N}(T_i)\} \{\Delta_j^* - F(T_j)\} \\
&\quad \cdot \int_{t=a}^b \frac{g_1(t)K_{b_N}(t-T_i)K_{b_N}(t-T_j)}{g_2(t)\bar{g}_N(t)\tilde{F}_{N,\tilde{b}_N}(t)\{1-\tilde{F}_{N,\tilde{b}_N}(t)\}} dt, \\
C_N^* &= \frac{2\alpha_N\beta_N}{mn} \sum_{i=1}^m \sum_{j=m+1}^N \{\Delta_i^* - \tilde{F}_{N,\tilde{b}_N}(T_i)\} \{\Delta_j^* - \tilde{F}_{N,\tilde{b}_N}(T_j)\} \\
&\quad \cdot \int_{t=a}^b \frac{K_{b_N}(t-T_i)K_{b_N}(t-T_j)}{\bar{g}_N(t)\tilde{F}_{N,\tilde{b}_N}(t)\{1-\tilde{F}_{N,\tilde{b}_N}(t)\}} dt,
\end{aligned}$$

and the bias term D_N is given by:

$$D_N = \alpha_N\beta_N \int_{t=a}^b \frac{f(t)^2 \{g_1'(t)g_2(t) - g_2'(t)g_1(t)\}^2}{F(t)\{1-F(t)\}\bar{g}_N(t)g_1(t)g_2(t)} dt \left\{ \int u^2 K(u) du \right\}^2 b_N^4.$$

Note (again) that $D_N = 0$ if $g_1 = g_2$.

However, the distribution function $\tilde{F}_{N,\tilde{b}_N}$ does not satisfy the condition that the second derivative is uniformly bounded on an interval (a', b') , containing $[a, b]$, which is a condition on F in Theorems 2.1 to 2.3. But a scrutiny of the proof of Lemma 8.2 reveals that this condition was only needed to take care of the bias term

$$\begin{aligned}
&n^{-1} \sum_{i=m+1}^N K_{b_N}(t-T_i) m^{-1} \sum_{i=1}^m \{F(T_i) - F(t)\} K_{b_N}(t-T_i) \\
&\quad - m^{-1} \sum_{i=1}^m K_{b_N}(t-T_i) n^{-1} \sum_{i=m+1}^n \{F(T_i) - F(t)\} K_{b_N}(t-T_i),
\end{aligned}$$

see (8.10), which in the present case transforms into

$$\begin{aligned}
&n^{-1} \sum_{i=m+1}^N K_{b_N}(t-T_i) m^{-1} \sum_{i=1}^m \{\tilde{F}_{N,\tilde{b}_N}(T_i) - \tilde{F}_{N,\tilde{b}_N}(t)\} K_{b_N}(t-T_i) \\
&\quad - m^{-1} \sum_{i=1}^m K_{b_N}(t-T_i) n^{-1} \sum_{i=m+1}^n \{\tilde{F}_{N,\tilde{b}_N}(T_i) - \tilde{F}_{N,\tilde{b}_N}(t)\} K_{b_N}(t-T_i),
\end{aligned}$$

since we do not change the T_i of the original samples. But since

$$\int \delta K_{\tilde{b}_N}''(t-u) d\mathbb{P}_N(u, \delta) = \frac{1}{N\tilde{b}_N^3} \sum_{i=1}^N K''\left(\frac{t-T_i}{\tilde{b}_N}\right) \Delta_i = O_p\left(\sqrt{\log N}\right),$$

uniformly in $t \in [a, b]$, using again the methods of Lemma 8.1 together with the assumption that F , g_1 and g_2 are twice continuously differentiable, the remainder term $O(b_N^4)$ in (8.10) can be replaced by a remainder term of order $O_p(b_N^4 \log N)$, which is sufficient for our purposes. Theorem 4.1 now follows. \square

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